

# POINTWISE EIGENFUNCTION ESTIMATES AND INTRINSIC ULTRACONTRACTIVITY-TYPE PROPERTIES OF FEYNMAN-KAC SEMIGROUPS FOR A CLASS OF LÉVY PROCESSES

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**ABSTRACT.** We introduce a class of Lévy processes subject to specific regularity conditions, and consider their Feynman-Kac semigroups given under a Kato-class potential. First we analyze intrinsic ultracontractivity and related properties of the semigroup refining them by the concept of ground state domination and asymptotic versions. We establish the relationships of these properties, derive sharp necessary and sufficient conditions for their validity in terms of the behaviour of the Lévy density and the potential at infinity, define the concept of borderline potential for the asymptotic properties, and give probabilistic and variational characterizations. Next we analyze the rate of decay of eigenfunctions at infinity. We prove bounds on  $\lambda$ -subaveraging functions, from which we derive two-sided sharp pointwise estimates on the ground state, and obtain upper bounds on all other eigenfunctions. These results are amply illustrated by key examples.

*Key-words:* Lévy process, subordinate Brownian motion, Feynman-Kac semigroup, non-local operator, intrinsic ultracontractivity, entropy, ground state domination, decay of eigenfunctions,  $\lambda$ -subaveraging functions

2010 *MS Classification:* Primary 47D08, 60G51; Secondary 47D03, 47G20

## 1. Introduction

Jump Lévy processes differ in a number of essential ways from Brownian motion. In this paper we focus on two aspects of this qualitatively different behaviour under the effect of a potential (or penalty function) on the paths. One is a strong smoothing property of the semigroup of such a process called intrinsic ultracontractivity. The other is the rate of decay of its eigenfunctions. These two properties are related and in this paper we will discuss the extent of this relationship.

We consider a class of symmetric jump Lévy processes satisfying specific regularity conditions. One condition is given in terms of the convolution of their jump intensities by a restriction to a subset of the full state space in relation to large jumps. Another is existence of transition densities and their uniform boundedness in space after at least a sufficiently long time. A final condition requires sufficient regularity of the Green function for specially chosen balls. These conditions are formulated in Assumptions 2.1–2.3 below. As it will be seen, they are satisfied by important classes of Lévy processes, including many cases of interest such as rotationally symmetric stable and relativistic stable, mixed stable, geometric stable, jump-diffusion, symmetric stable-like processes, and others.

Next we introduce a potential function  $V$  and study the Lévy processes perturbed by it in terms of their Feynman-Kac semigroups  $\{T_t : t \geq 0\}$ . Under a suitable choice of  $V$ , which we call  $X$ -Kato class associated with Lévy process  $(X_t)_{t \geq 0}$  (see Definition 2.1 below), the semigroup  $\{T_t : t \geq 0\}$  is well defined and consists of symmetric operators. We additionally assume that  $V(x) \rightarrow \infty$  as  $|x| \rightarrow \infty$ , which implies that all  $T_t$  are compact and have a discrete spectrum. The corresponding

eigenfunctions  $\varphi_n$  form an orthonormal basis in  $L^2(\mathbf{R}^d, dx)$  and satisfy  $T_t \varphi_n = e^{-\lambda_n t} \varphi_n$ , where  $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \rightarrow \infty$ . All  $\varphi_n$  are bounded continuous functions, and each  $\lambda_n$  has finite multiplicity. We call  $\varphi_0$  ground state, which can be shown to be unique and strictly positive.

Since a ground state  $\varphi_0$  exists, we can define the intrinsic Feynman-Kac semigroup

$$(1.1) \quad \tilde{T}_t f(x) = \frac{e^{\lambda_0 t}}{\varphi_0(x)} T_t(\varphi_0 f)(x)$$

which is a Markov semigroup on  $L^2(\mathbf{R}^d, \varphi_0^2 dx)$ . Whenever  $T_t$  has an integral kernel  $u(t, x, y)$ , the operators  $\tilde{T}_t$  are given by the kernels

$$(1.2) \quad \tilde{u}(t, x, y) = \frac{e^{\lambda_0 t} u(t, x, y)}{\varphi_0(x) \varphi_0(y)}.$$

The intrinsic Feynman-Kac semigroup  $\{\tilde{T}_t : t \geq 0\}$  describes a stationary Markov process which is called  $P(\phi)_1$ -process associated with potential  $V$  [75, 9, 47].

Intrinsic ultracontractivity (IUC) means that  $\tilde{T}_t$  is a bounded operator from  $L^2(\mathbf{R}^d, \varphi_0^2 dx)$  to  $L^\infty(\mathbf{R}^d)$  for every  $t > 0$ , or equivalently,  $\tilde{u}(t, x, y) \leq C$  for all  $x, y \in \mathbf{R}^d$ , with an appropriate constant  $C$  dependent on  $V$  and  $t$ . IUC has been first introduced in [32] for general semigroups of compact operators and it proved to be a strong regularity property [31]. Important examples include semigroups of elliptic operators and Schrödinger semigroups on function spaces over  $\mathbf{R}^d$  or bounded domains of  $\mathbf{R}^d$  with Dirichlet boundary conditions [3, 31, 4, 48, 1, 62, 77]. More recently, IUC has been addressed also in the case of semigroups generated by fractional Laplacians, and fractional and relativistic Schrödinger operators [27, 28, 55, 57, 58, 46, 47], as well as for more general symmetric [37] and non-symmetric [49] Lévy processes in bounded domains. In the context of parabolic partial differential equations [64, 65] obtained integral representations of the non-negative solutions of the Cauchy problem when intrinsic ultracontractivity holds.

As it follows from our previous analysis, it is of interest to consider also the property that  $\tilde{u}(t, x, y) \leq C$ , for all  $x, y \in \mathbf{R}^d$  and sufficiently large  $t$  only, which we call asymptotic intrinsic ultracontractivity (AIUC). This is weaker than IUC, and we have seen in the case of fractional  $P(\phi)_1$ -processes [47] that it has an immediate impact on the support properties of their (Gibbs) path measure. One important consequence is that AIUC is equivalent to

$$\left| \frac{e^{\lambda_0 t} u(t, x, y)}{\varphi_0(x) \varphi_0(y)} - 1 \right| \leq C e^{-(\lambda_1 - \lambda_0)t}, \quad t > t_0,$$

which means that the distribution of the corresponding  $P(\phi)_1$  process rapidly tends to equilibrium as  $t \rightarrow \infty$  with decay rate given by the spectral gap  $\lambda_1 - \lambda_0$ . This, in turn, has an offshoot on the efficiency of practical sampling of conditioned processes, see e.g. [39, 40]. Also, it implies that the kernel  $u(t, x, y)$  takes the shape of the ground state exponentially quickly, in particular, the decay of the eigenfunctions  $\varphi_1, \varphi_2, \dots$  will be dominated by the decay of the ground state.

A basic question we address in this paper is that given the class of jump Lévy processes considered, what are conditions on  $V$  making the Feynman-Kac semigroup  $\{T_t : t \geq 0\}$  IUC or AIUC. The answer is, roughly, that this is decided by how the asymptotic behaviours of  $V$  and  $|\log \nu|$  at infinity compare, where  $\nu$  is the Lévy density. We further refine IUC-type properties by considering a ground state domination (GSD) property and its asymptotic version for sufficiently long times (AGSD). We clarify the relationships of these properties (Theorem 2.5), and give precise necessary and sufficient conditions (Theorems 2.6 and 2.7) for them to hold. Our results recover the known facts for stable processes [46, 47] and relativistic stable processes [57], and give a unified characterization generalizing far beyond these only available results, also shedding new light on existing results

for diffusions [32, 31]. Corollary 2.3 gives a sharp description of the borderline potential  $V(x) \asymp |\log \nu(x)|$  distinguishing (A)IUC from non-(A)IUC. In comparison with the classic result which says, roughly, that IUC holds for a diffusion when  $V$  grows in leading order super-quadratically, it is seen that for a jump Lévy process it is “easier” to be (A)IUC than for a diffusion. We give an explanation of this in terms of a balance mechanism between the competition of killing and survival of paths (Proposition 2.2), and give a probabilistic characterization (Propositions 2.3 and 2.4). Furthermore (Theorem 2.8, Corollary 2.4), we obtain a second characterization of AIUC in terms of minimizing a free energy functional appearing as the difference of an energy and an entropy associated with the Lévy measure (see (2.11)-(2.13) below), and obtain the borderline potential as the solution of this variational problem. Due to the role played by the entropy this also explains why  $\log \nu$  appears in this expression.

A second basic problem we address is to derive pointwise bounds on the eigenfunctions for a given Lévy process and  $V$ . We obtain sharp lower and upper bounds (Theorem 2.4) showing that the fall-off of the ground state follows the tail behaviour of the Lévy density with corrections resulting from the contribution of the potential. Furthermore, we obtain upper bounds on all other eigenfunctions (Theorem 2.3, Corollary 2.1). Under a reasonable condition we derive a more explicit expression of the dependence of the decay on  $V$  (Corollary 2.2). We note that, importantly, the ground state estimates follow without any need to use results on the (A)IUC properties, unlike in the previous work [57]. Our considerations lead naturally to defining  $\lambda$ -subaveraging functions as counterparts to  $\lambda$ -superaveraging functions known in potential theory, and we prove two results on them (Theorems 2.1 and 2.2). Although it makes the paper longer, we find it useful to discuss important (classes of) examples in relation to both the ground state bounds and the IUC-type properties. We also discuss in some detail which types of specific cases are covered by the Lévy processes we tackle in this paper, as well as interesting cases which fail the assumptions or the IUC-properties.

We note that our results can also be considered from the perspective of the correspondence between jump Lévy processes and non-local operators, which are their generators. Indeed, via a Feynman-Kac representation our results are as well about the decay of eigenfunctions and IUC-type properties of semigroups related to generalized Schrödinger operators whose kinetic terms are given by the generators of these processes. A specific class of operators covered to a large extent by our results are of the form  $\Psi(-\Delta) + V$ , recently studied in [41, 42], where  $\Psi$  is a Bernstein function with vanishing right limit at zero (i.e., no killing term), giving the Lévy exponent of a subordinator [6, 73]. These operators correspond to subordinate Brownian motions under potential  $V$ , and include important cases of non-local operators like fractional Schrödinger operators  $(-\Delta)^{\alpha/2} + V$  (see Examples 2.2(i) and 2.3(1) below), relativistic Schrödinger operators  $(-\Delta + m^{2/\alpha})^{\alpha/2} - m + V$  (Examples 2.2(iv) and 2.3(3)), jump-diffusion operators  $a(-\Delta)^{\alpha/2} - b\Delta + V$  (Examples 2.2(iii) and 2.3(4)) and many others. Some of these operators have been widely studied from both a probabilistic and an analytic point of view [18, 11, 12, 76, 26, 57, 34, 59, 46, 47, 60, 45, 20, 5, 56].

The basic input object in this paper is a Lévy process, therefore we mainly use probabilistic methods. Roughly speaking, these methods build on a combination of sharp uniform estimates on the local extrema of functions harmonic with respect to the sub-process obtained under the Feynman-Kac functional of the Lévy process (Lemma 3.1) developed only recently in [14] (see also [51]), and powerful enough estimates of the expectations of this functional with respect to the path measure of the process (see the proofs of Theorems 2.1 and 3.1). Estimates of harmonic functions were previously used in a similar context in the case of rotationally symmetric stable processes in [58, 46, 47], however, the methods there are tied to this class of processes and cannot be extended directly. The upper bound on the expectation of the Feynman-Kac functional is proved by extending

a jump estimation scheme (see eqs. (3.11), (3.12) and (3.21) below) which was originally devised in [17, 57] for rotationally symmetric stable and relativistic stable processes. To make use of it in our more general framework we need a substantial step, including strengthened estimates of jumps of the process between some specially chosen annular sets (see Subsection 3.2) generalizing and improving those in [57] essentially. This will be achieved by using a new technique based on the uniform estimates of harmonic functions, and by introducing and exploiting the pivotal Assumption 2.1(3), which turns out to be very general and rather natural. Since IUC has been much studied in operator semigroup and PDE context, we find it important to develop a probabilistic understanding of it, and our methods serve this purpose well.

The remainder of the paper is organized as follows. In Section 2 we state the main results. First we introduce the class of Lévy processes which will be considered and test the conditions on examples. The next two subsections are devoted to presenting the estimates on  $\lambda$ -subaveraging functions, ground states and the higher order eigenfunctions. The last two subsections present the results on intrinsic ultracontractivity and ground state domination, including examples. In Section 3 we provide the proofs in a similar division of the material by the topics.

## 2. Main results

### 2.1. A class of Lévy processes

Let  $(X_t)_{t \geq 0}$  be a Lévy process in  $\mathbf{R}^d$ ,  $d \geq 1$ . We use the notations  $\mathbf{P}^x$  and  $\mathbf{E}^x$ , respectively, for the probability measure and expected value of the process starting in  $x \in \mathbf{R}^d$ . Recall that  $(X_t)_{t \geq 0}$  is a Markov process with respect to its own filtration satisfying the strong Markov property, and has right continuous paths with left limits (càdlàg paths). It is a basic fact that  $(X_t)_{t \geq 0}$  is completely determined by its characteristic exponent  $\psi$  given by the Lévy-Khintchine formula, i.e., for  $\xi \in \mathbf{R}^d$

$$\mathbf{E}^0 \left[ e^{i\xi \cdot X_t} \right] = e^{-t\psi(\xi)}$$

holds with

$$\psi(\xi) = -i\gamma \cdot \xi + A\xi \cdot \xi + \int_{\mathbf{R}^d} \left( 1 - e^{i\xi \cdot z} + i\xi \cdot z \mathbf{1}_{\{|z| < 1\}}(z) \right) \nu(dz).$$

Here  $\gamma \in \mathbf{R}^d$  (*drift coefficient*),  $A$  is a symmetric non-negative definite  $d \times d$  matrix (*diffusion* or *Gaussian coefficient*), and  $\nu$  is a Radon measure on  $\mathbf{R}^d \setminus \{0\}$  such that  $\int_{\mathbf{R}^d} (1 \wedge |z|^2) \nu(dz) < \infty$  (*Lévy measure*). The defining parameters  $(\gamma, A, \nu)$  are called the *Lévy triplet* of the process.  $(X_t)_{t \geq 0}$  is said to be *symmetric* whenever  $X_t$  has the same distribution as  $-X_t$  for all  $t > 0$ . In this case  $\psi(\xi) = \psi(-\xi)$ ,  $\xi \in \mathbf{R}^d$ , i.e.,  $\gamma \equiv 0$  and  $\nu(B) = \nu(-B)$  for all  $B \in \mathcal{B}(\mathbf{R}^d \setminus \{0\})$ , and then the characteristic exponent reduces to

$$\psi(\xi) = A\xi \cdot \xi + \int_{\mathbf{R}^d} (1 - \cos(\xi \cdot z)) \nu(dz).$$

Whenever  $\nu(dz)$  is absolutely continuous with respect to Lebesgue measure we denote its density by  $\nu(z)$  and call it the *Lévy (jump) intensity* of  $(X_t)_{t \geq 0}$ . When  $A \equiv 0$  and  $\nu \neq 0$ , the process  $(X_t)_{t \geq 0}$  is said to be a *purely jump* Lévy process. For more details on Lévy processes we refer to [7, 2, 72, 35, 44].

We will use throughout the notation  $C(a, b, c, \dots)$  for a positive constant dependent on parameters  $a, b, c, \dots$ , while dependence on the process  $(X_t)_{t \geq 0}$  is indicated by  $C(X)$ , and dependence on the dimension  $d$  is assumed without being indicated. Since constants appearing in definitions, lemmas and theorems play a role later on, we use the numbering  $C_1, C_2, \dots$  to be able to track them. We

will also use the notation  $f \asymp Cg$  meaning that  $C^{-1}g \leq f \leq Cg$  with a constant  $C$ , while  $f \asymp g$  means that there is a constant  $C$  such that the latter holds.

For the remainder of the paper we make three standing assumptions on the Lévy processes we consider.

**Assumption 2.1.**  $(X_t)_{t \geq 0}$  is a symmetric Lévy process with Lévy measure absolutely continuous with respect to Lebesgue measure. The corresponding jump intensity  $\nu(x)$  is strictly positive and satisfies the following three conditions:

- (1) For every  $0 < r \leq 1/2$  there is a constant  $C_1 = C_1(X, r) \geq 1$  such that

$$\nu(x) \asymp C_1 \nu(y), \quad r \leq |y| \leq |x| \leq |y| + 1.$$

- (2) There is a constant  $C_2 = C_2(X) \geq 1$  such that

$$\nu(x) \leq C_2 \nu(y), \quad 1/2 \leq |y| \leq |x|.$$

- (3) There is a constant  $C_3 = C_3(X) \geq 1$  such that

$$\int_{|z-x| > 1/2, |z-y| > 1/2} \nu(x-z)\nu(z-y)dz \leq C_3 \nu(x-y), \quad |y-x| \geq 1.$$

Conditions (1) and (2) are clearly satisfied when e.g.  $\nu(x) \asymp \kappa(|x|)$ ,  $x \in \mathbf{R}^d$ , where  $\kappa : (0, \infty) \rightarrow (0, \infty)$  is a non-increasing function such that  $\kappa(s) \leq C\kappa(s+1)$ ,  $s \geq 1/2$ . Condition (3) provides a regularity of the convolutions of  $\nu$  with respect to large jumps. While (1)-(2) can be seen as technical conditions, (3) has a structural importance.

**Example 2.1.** We illustrate the assumption above by some classes of examples of interest. The first three examples satisfy conditions (1)-(3) of Assumption 2.1, the last ones are counterexamples.

- (1) Let  $\nu(x) \asymp |x|^{-d-\alpha}(1+|x|)^{\alpha-\beta}$ ,  $x \in \mathbf{R}^d$ , for  $\alpha \in [0, 2)$  and  $\beta > 0$ . This class includes a large family of symmetric Lévy processes. The most studied cases of this class are listed in Example 2.2 (1)(i)-(iii), (v) and (3) below.
- (2) Also, if  $\nu(x) \asymp \kappa(|x|)|x|^{-d-\alpha}$ ,  $\alpha \in (0, 2)$ , where  $\kappa : [0, \infty) \rightarrow (0, 1]$  is a non-increasing function such that  $\kappa(0) = 1$  and  $\kappa(a)\kappa(b) \leq C\kappa(a+b)$ ,  $a, b, C > 0$ , then all conditions on  $\nu$  are verified directly. Examples include  $\kappa(s) = 1/\log(e+s)$  and  $\kappa(s) = 1/(\log(e+\log(1+s)))$ .
- (3) A case of special interest is  $\nu(x) \asymp e^{-a|x|^\beta}|x|^{-d-\delta}(1+|x|)^{d+\delta-\gamma}$  with  $a > 0$ ,  $\beta > 0$ ,  $\delta \in [0, 2)$  and  $\gamma > 0$  (see Example 2.2 (4)). In this case condition (2) always holds, condition (1) is satisfied when  $\beta \in (0, 1]$  without further restriction, and condition (3) is satisfied when moreover  $\gamma > (d+1)/2$ . This case covers the important class of rotationally symmetric relativistic  $\alpha$ -stable processes ( $a = m^{1/\alpha}$ ,  $\beta = 1$ ,  $\gamma = (d+\delta+1)/2$ , with  $m > 0$  and  $\delta \in (0, 2)$ ) and (exponentially) tempered symmetric stable processes ( $a > 0$ ,  $\beta = 1$ ,  $\gamma = \delta+d$ ,  $\delta \in (0, 2)$ ), see Examples 2.2 (1iv) and (4).
- (4) For  $\beta > 1$  in case (3) above condition (3) of the assumption is not satisfied. Similarly, at least in one dimension, it fails when  $\beta = 1$  and  $\gamma = (d+1)/2$ . Note that while Assumption 2.1 covers many important processes that appear through subordination of standard Brownian motion [13], it is not true that every subordinate Brownian motion satisfies it. A counterexample is the one-dimensional geometric 2-stable process (see Example 2.2 (1)(vi) below). In this case  $\nu(x) \asymp e^{-|x|}|x|^{-1}$ ,  $|x| > 1/2$ , and condition (3) of Assumption 2.1 fails.

Denote by  $P_t f(x) = \mathbf{E}^x f(X_t)$  the transition operators of the process  $(X_t)_{t \geq 0}$ . Recall that  $(X_t)_{t \geq 0}$  has the strong Feller property if  $P_t(L^\infty(\mathbf{R}^d)) \subset C_b(\mathbf{R}^d)$ , for all  $t > 0$ .

**Assumption 2.2.** *The process  $(X_t)_{t \geq 0}$  has the strong Feller property, or equivalently, the one-dimensional  $\mathbf{P}^x$ -distributions of  $(X_t)_{t \geq 0}$  are absolutely continuous with respect to Lebesgue measure, i.e., there exist transition probability densities  $p(t, x, y) = p(t, y - x, 0) =: p(t, y - x)$ . Furthermore, there exist  $t_b > 0$  and  $C_4 = C_4(X, t_b)$  such that  $0 < p(t_b, x) \leq C_4$ , for all  $x \in \mathbf{R}^d$ .*

Note that the first part of the above assumption is satisfied by a large class of Lévy processes including subordinate Brownian motion [50] provided that they are not compound Poisson processes. Boundedness of the density is equivalent to  $e^{-t_b \psi(\cdot)} \in L^1(\mathbf{R}^d)$  for some  $t_b > 0$ . In this case  $p(t_b, x)$  can be obtained by the Fourier inversion formula. Clearly, this property extends to all  $t \geq t_b$  by the Markov property of  $(X_t)_{t \geq 0}$ . For more details on existence of transition probability densities for Lévy processes we refer to [54] and references therein.

We note for later use that under Assumption 2.2 transition densities  $p_D(t, x, y)$  of the process  $(X_t)_{t \geq 0}$  killed upon exiting an open bounded set  $D \subset \mathbf{R}^d$  also exist. In this case the Hunt formula

$$(2.1) \quad p_D(t, x, y) = p(t, y - x) - \mathbf{E}^x [\tau_D < t; p(t - \tau_D, y - X_{\tau_D})], \quad x, y \in D,$$

holds, where  $\tau_D = \inf \{t \geq 0 : X_t \notin D\}$  is the first exit time from  $D$ . The Green function of the process  $(X_t)_{t \geq 0}$  on  $D$  is thus given by  $G_D(x, y) = \int_0^\infty p_D(t, x, y) dy$ , for all  $x, y \in D$ .

Since our results rely on a use of potential theory, we need some more regularity of  $(X_t)_{t \geq 0}$ .

**Assumption 2.3.** *For all  $0 < p < q < R \leq 1$  we have  $\sup_{x \in B(0, p)} \sup_{y \in B(0, q)^c} G_{B(0, R)}(x, y) < \infty$ .*

In many cases Assumption 2.3 is a direct consequence of time-space estimates of the function  $p(t, x)$ . Indeed, it is clearly satisfied when the boundedness condition holds with  $G_{B(0, R)}(x, y)$  replaced by the potential kernel  $G_{\mathbf{R}^d}(x, y) = \int_0^\infty p(t, x, y) dt$  or, as proved in [14, Proposition 2.3], the  $\lambda$ -potential kernel  $G_{\mathbf{R}^d}^\lambda(x, y) = \int_0^\infty e^{-\lambda t} p(t, x, y) dt$ ,  $\lambda > 0$ , whenever the process  $(X_t)_{t \geq 0}$  is recurrent.

One of our key arguments following below uses some estimates (see Lemma 3.1) on the local extrema of functions harmonic with respect to the sub-process of  $(X_t)_{t \geq 0}$  obtained under its Feynman-Kac functional. These bounds are a direct consequence of more general results of Bogdan, Kumagai and Kwaśnicki obtained recently in [14]. To borrow these results, we need to match some assumptions made in this paper, however, since we consider symmetric Lévy processes, Assumptions 2.1 (1), 2.2 (without requiring boundedness of  $p(t, x)$ ) and 2.3 provide sufficient regularity of  $(X_t)_{t \geq 0}$  to allow a use of [14]. The remaining conditions in Assumption 2.1 are independent from this context and together with condition (1) for  $r = 1/2$  only they will allow to draw more regularity of the Lévy intensity  $\nu$  needed in controlling jumps in Subsection 3.2 below. Similarly, boundedness of  $p(t, x)$  in Assumption 2.2 guarantees sufficient regularity of the process needed below.

We now test these assumptions on some important specific classes of Lévy processes.

### Example 2.2.

- (1) *Subordinate Brownian motions* with characteristic exponents  $\psi$  such that  $e^{-t_b \psi(\cdot)} \in L^1(\mathbf{R}^d)$  for some  $t_b > 0$ , whenever their Lévy measures satisfy Assumption 2.1. Since in this case  $\nu(x)$  is radially decreasing, condition (2) of Assumption 2.1 is automatically satisfied, however, not necessarily the remaining conditions (1) and (3). Condition (1) is always satisfied as long as  $\nu(x) \leq C\nu(y)$  for all  $|x| \geq 1$ ,  $|y| = |x| + 1$ , and a discussion of condition (3) has already been made in Example 2.1. The transition densities  $p(t, y - x)$  are given by the subordination formula, i.e., by the integral over time of the Brownian transition kernel with respect to the distribution of the given subordinator. Since also  $e^{-t_b \psi(\cdot)}$  is integrable, Assumption 2.2 is satisfied (in particular,  $(X_t)_{t \geq 0}$  is not a compound Poisson process). Lastly, Assumption 2.3 follows from a similar bound for the potential or  $\lambda$ -potential kernel,

which is again a consequence of the subordination formula and an easy estimate. Below, (i)-(iv) are specific examples of subordinate Brownian motion of special interest satisfying all of our assumptions, and (v)-(vii) are examples where the situation is more complicated. For properties of subordinate Brownian motion and further examples see [13, 50, 73, 8].

- (i) *Rotationally symmetric  $\alpha$ -stable process.* Let  $\psi(\xi) = |\xi|^\alpha$ ,  $\alpha \in (0, 2)$ . In this case  $\nu(x) = C(\alpha)|x|^{-d-\alpha}$ .
- (ii) *Mixture of independent rotationally symmetric stable processes with indices  $\alpha$  and  $\beta$ .* This is obtained for  $\psi(\xi) = a|\xi|^\alpha + b|\xi|^\beta$ ,  $0 < \beta < \alpha < 2$ ,  $a, b > 0$ . We have  $\nu(x) = aC(\alpha)|x|^{-d-\alpha} + bC(\beta)|x|^{-d-\beta}$ .
- (iii) *Jump-diffusion process* [22, 24]. Let  $\psi(\xi) = a|\xi|^\alpha + b|\xi|^2$ ,  $0 < \alpha < 2$ ,  $a, b > 0$ , i.e., the process is a mixture of a rotationally symmetric  $\alpha$ -stable process and an independent Brownian motion. In this case  $\nu(x) \asymp aC(\alpha)|x|^{-d-\alpha}$ .
- (iv) *Relativistic rotationally symmetric  $\alpha$ -stable process* [70, 21]. Let  $\psi(\xi) = (|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m$ ,  $\alpha \in (0, 2)$ ,  $m > 0$ . It is known that  $\nu(x) \asymp e^{-m^{1/\alpha}|x|}|x|^{-d-\alpha}(1 + |x|^{(d+\alpha-1)/2})$  (see e.g. [57]).
- (v) *Rotationally symmetric geometric  $\alpha$ -stable process* [74, 38]. Let  $\psi(\xi) = \log(1 + |\xi|^\alpha)$ ,  $0 < \alpha < 2$ . In this case  $\nu(x) \asymp |x|^{-d}(1 + |x|)^{-\alpha}$ . Notice that  $\psi$  is now a slowly varying function at infinity. In contrast to the previous examples, in this case there is  $t(\alpha) > 0$  for which the transition probability densities are unbounded for  $0 < t < t(\alpha)$  [13, p. 117], though they are bounded for large  $t$ .
- (vi) *Rotationally symmetric geometric 2-stable (or gamma variance) process.* Let  $\psi(\xi) = \log(1 + |\xi|^2)$ . In this case  $\nu(x) \asymp |x|^{-d}e^{-|x|}(1 + |x|)^{(d-1)/2}$ . As in Example 2.1 (4), at least in dimension one, the Lévy measure does not satisfy condition (3) of Assumption 2.1.
- (vii) *Iterated rotationally symmetric geometric  $\alpha$ -stable process.* Let  $\psi(\xi) = \log(1 + \log^\alpha(1 + |\xi|^\alpha))$ ,  $0 < \alpha < 2$ . It can be checked directly that the transition densities are unbounded for any  $t > 0$  (see [13, p. 117]) and the second part of Assumption 2.2 fails.
- (2) *Symmetric Lévy processes with non-degenerate Brownian part* [52]. Let  $(X_t)_{t \geq 0}$  be a Lévy process with characteristic exponent  $\psi(\xi) = c|\xi|^2 + \int_{\mathbf{R}^d} (1 - \cos(z \cdot \xi))\nu(dz)$ ,  $c > 0$ , i.e., a sum of Brownian motion with rescaled time  $(B_{2ct})_{t \geq 0}$  and an independent symmetric Lévy process  $(Y_t)_{t \geq 0}$  with Lévy measure  $\nu$  satisfying Assumption 2.1. In this case the transition densities are given by the convolution of the Gaussian kernel and the distribution of the process  $(Y_t)_{t \geq 0}$  (note that we do not need to assume that  $(Y_t)_{t \geq 0}$  has transition densities) or by the Fourier inversion formula, and they are clearly bounded for all  $t > 0$ . Thus Assumption 2.2 also is satisfied. In one dimension Assumption 2.3 easily follows from a similar bound for the corresponding  $\lambda$ -potential kernel. In higher dimensions, the required upper bound for  $\lambda$ -potential kernel can be proved by showing that, for instance, for every  $\varepsilon > 0$  there is  $t_\varepsilon > 0$  such that

$$(2.2) \quad \mathbf{P}^0(Y_t \in A) \leq C|A|, \quad \text{whenever } A \subset \mathcal{B}(\mathbf{R}^d), \text{ dist}(0, A) > \varepsilon, t \in (0, t_\varepsilon),$$

(here  $|A|$  denotes Lebesgue measure of  $A$ ) with constant  $C = C(Y, \varepsilon)$  independent of  $t$  and the specific  $A$ . Specific cases are jump-diffusions as above and many similar processes in which the rotationally symmetric stable process is replaced by other symmetric Lévy processes  $(Y_t)_{t \geq 0}$  satisfying Assumption 2.1 and condition (2.2).

- (3) *Symmetric stable-like Lévy processes* [23]. Let  $\alpha \in (0, 2)$  and  $(X_t)_{t \geq 0}$  be a purely jump (i.e., with no diffusion part) symmetric Lévy process with Lévy intensity  $\nu(x) \asymp C|x|^{-d-\alpha}$ ,  $x \in \mathbf{R}^d$ . It is known [23] that  $(X_t)_{t \geq 0}$  has bounded continuous transition probability densities such that  $p(t, y - x) \asymp t^{-d/\alpha} \wedge t|y - x|^{-d-\alpha}$ ,  $t > 0$ ,  $x, y \in \mathbf{R}^d$ . In this case Assumptions 2.1 and 2.2 are clearly satisfied, while Assumption 2.3 is an easy consequence of a similar bound for the potential kernel ( $\alpha < d$ ) or the  $\lambda$ -potential kernel ( $1 = d \leq \alpha < 2$ ) of  $(X_t)_{t \geq 0}$ . This class includes a subclass of strictly stable Lévy processes with intensities of the form  $|x|^{-d-\alpha}f(x/|x|)$  with functions  $f(x) = f(-x)$  that are bounded from above and below by positive constants [15].
- (4) *Symmetric Lévy processes with sub-exponentially localized Lévy measures*. Let  $(X_t)_{t \geq 0}$  be a symmetric Lévy process with intensity  $\nu(x) \asymp e^{-a|x|^\beta}|x|^{-d-\delta}(1+|x|)^{d+\delta-\gamma}$ , where  $a > 0$ ,  $\beta \in (0, 1]$ ,  $\delta \in [0, 2)$  and  $\gamma > (d+1)/2$ . Such processes were considered in more general settings in [19] (see also [25]) and references therein. As discussed in Example 2.1 (3), Assumption 2.1 is verified. Moreover, as proved in a greater generality in [19, Theorem 1.2 (1)],  $(X_t)_{t \geq 0}$  is a strong Feller process with bounded continuous transition densities satisfying appropriate sharp two-sided bounds with respect to large and small times separately (see [19, (1.13) and (1.14)]). Thus also Assumption 2.2 holds. As before, the required bound on the Green function in Assumption 2.3 may be obtained by showing the same estimate for the  $\lambda$ -potential kernel of the process  $(X_t)_{t \geq 0}$ , which can be easily done by using the sharp transition density estimates referred to above. This class includes a large family of (exponentially) tempered symmetric stable processes [71] and the rotationally symmetric relativistic stable processes above.

Next we give the class of potentials which will be used in this paper.

**Definition 2.1** (*X-Kato class*). We say that the Borel function  $V : \mathbf{R}^d \rightarrow \mathbf{R}$  belongs to the Kato-class  $\mathcal{K}^X$  associated with the Lévy process  $(X_t)_{t \geq 0}$  if it satisfies

$$(2.3) \quad \lim_{t \rightarrow 0} \sup_{x \in \mathbf{R}^d} \mathbf{E}^x \left[ \int_0^t |V(X_s)| ds \right] = 0.$$

We write  $V \in \mathcal{K}_{\text{loc}}^X$  if  $V \mathbf{1}_B \in \mathcal{K}^X$  for every ball  $B \subset \mathbf{R}^d$ . Moreover, we say that  $V$  is an *X-Kato decomposable potential* whenever

$$V = V_+ - V_- \quad \text{with} \quad V_- \in \mathcal{K}^X, \quad V_+ \in \mathcal{K}_{\text{loc}}^X,$$

where  $V_+$  and  $V_-$  denote the positive and negative parts of  $V$ , respectively.

For simplicity, in what follows we refer to *X-Kato decomposable potentials* as *X-Kato class potentials*. It is easy to see that  $L_{\text{loc}}^\infty \subset \mathcal{K}_{\text{loc}}^X$ . Moreover, by stochastic continuity of  $(X_t)_{t \geq 0}$  also  $\mathcal{K}_{\text{loc}}^X \subset L_{\text{loc}}^1(\mathbf{R}^d)$ , and thus an *X-Kato class potential* is always locally absolutely integrable. Note that condition (2.3) allows local singularities of  $V$ . For specific processes  $(X_t)_{t \geq 0}$  condition (2.3) can be equivalently reformulated in terms of the potential kernel of the process in the transient case, and the so called compensated potential kernel when the process is recurrent (for more details see e.g. [78, 18, 12]).

We single out a restricted set of potentials which will be often used below.

**Assumption 2.4.** For a given Lévy process  $(X_t)_{t \geq 0}$  let  $V$  be such that

- (1)  $V$  is an *X-Kato class potential*
- (2)  $V(x) \rightarrow \infty$  as  $|x| \rightarrow \infty$ .



Next, for a given  $X$ -Kato class potential  $V$  we define

$$T_t f(x) = \mathbf{E}^x \left[ e^{-\int_0^t V(X_s) ds} f(X_t) \right], \quad f \in L^2(\mathbf{R}^d), \quad t > 0.$$

Using the Markov property and stochastic continuity of the process it can be shown that  $\{T_t : t \geq 0\}$  is a strongly continuous semigroup of symmetric operators on  $L^2(\mathbf{R}^d)$ , which we call the *Feynman-Kac semigroup* associated with the process  $(X_t)_{t \geq 0}$  and potential  $V$ . In particular, by the Hille-Yoshida theorem there exists a self-adjoint operator  $H$  bounded from below such that  $e^{-tH} = T_t$ . The operator  $H$  is often seen as a *generalized Schrödinger operator* based on the infinitesimal generator  $L$  of the process  $(X_t)_{t \geq 0}$ . Whenever  $V$  is relatively bounded with respect to  $L$  with relative bound less than 1 we can write  $H = -L + V$  as an operator sum.

We now summarize the basic properties of the operators  $T_t$ , some of which will be explicitly used below.

**Lemma 2.1.** *Let Assumptions 2.1–2.4 be satisfied. Then the following properties hold:*

- (1) *For all  $t > 0$ ,  $T_t$  are bounded operators on each  $L^p(\mathbf{R}^d)$ ,  $1 \leq p \leq \infty$ . The operators  $T_t : L^p(\mathbf{R}^d) \rightarrow L^p(\mathbf{R}^d)$  for  $1 \leq p \leq \infty$ ,  $t > 0$ ,  $T_t : L^p(\mathbf{R}^d) \rightarrow L^\infty(\mathbf{R}^d)$  for  $1 < p \leq \infty$ ,  $t \geq t_b$ , and  $T_t : L^1(\mathbf{R}^d) \rightarrow L^\infty(\mathbf{R}^d)$  for  $t \geq 2t_b$  are bounded.*
- (2) *For all  $t \geq 2t_b$ ,  $T_t$  has a bounded, measurable, and symmetric kernel  $u(t, \cdot, \cdot)$ , i.e.,  $T_t f(x) = \int_{\mathbf{R}^d} u(t, x, y) f(y) dy$ ,  $f \in L^p(\mathbf{R}^d)$ ,  $1 \leq p \leq \infty$ .*
- (3) *For all  $t > 0$  and  $f \in L^\infty(\mathbf{R}^d)$ ,  $T_t f(x)$  is a bounded continuous function.*
- (4) *For all  $t \geq 2t_b$  the operators  $T_t$  are positivity improving, i.e.,  $T_t f(x) > 0$  for all  $x \in \mathbf{R}^d$  and  $f \in L^2(\mathbf{R}^d)$  such that  $f \geq 0$  and  $f \neq 0$  a.e.*
- (5) *All operators  $T_t : L^2(\mathbf{R}^d) \rightarrow L^2(\mathbf{R}^d)$ ,  $t > 0$ , are compact.*

Properties (1)–(4) can be established by standard arguments based on [30, Section 3.2]. Property (5) is a consequence of  $V(x) \rightarrow \infty$  as  $|x| \rightarrow \infty$  and standard arguments based on approximation of  $T_t$ ,  $t \geq 2t_b$ , by compact operators [47, Lemma 3.2]. Clearly, compactness extends to all  $t > 0$  by the fact that  $T_t = e^{-tH}$  for a self-adjoint operator  $H$  and a use of the spectral theorem. Note that we do not assume that  $p(t, x)$  is bounded for all  $t > 0$ , and thus in general the operators  $T_t : L^p(\mathbf{R}^d) \rightarrow L^\infty(\mathbf{R}^d)$  need not be bounded for  $t < t_b$ .

The theory of operator semigroups implies that there exists an orthonormal basis in  $L^2(\mathbf{R}^d)$  consisting of the eigenfunctions  $\varphi_n$  given by  $T_t \varphi_n = e^{-\lambda_n t} \varphi_n$ ,  $t > 0$ ,  $n \geq 0$ , and  $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \rightarrow \infty$ . All  $\varphi_n$  are bounded continuous functions. Moreover, the first eigenfunction (or ground state)  $\varphi_0$  has a strictly positive version [69, Theorem XIII.43], which will be our choice throughout.

## 2.2. Estimates of $\lambda$ -subaveraging functions

Recall that one of the fundamental objects in potential theory are  $\lambda$ -*superaveraging* (and related  $\lambda$ -*excessive*) functions [29, Section 2.1]. For our purposes below it will be natural to consider  $\lambda$ -*subaveraging* functions, which in some sense are counterparts of  $\lambda$ -superaveraging functions in the opposite direction of domination. We say that a non-negative Borel function  $\varphi$  is  $\lambda$ -*subaveraging* for the semigroup  $\{T_t : t \geq 0\}$  with  $\lambda \geq 0$  if for every  $t > 0$  and  $x \in \mathbf{R}^d$ ,  $e^{\lambda t} T_t \varphi(x) \geq \varphi(x)$ .

For an open set  $D \subset \mathbf{R}^d$ , a Kato-class potential  $V$  and a non-negative or bounded Borel function  $\varphi$  we define the  $V$ -*Green operator* for the semigroup  $\{T_t : t \geq 0\}$  and set  $D$  (see Subsection 3.1),

$$G_D^V \varphi(x) = \mathbf{E}^x \left[ \int_0^{\tau_D} e^{-\int_0^t V(X_s) ds} \varphi(X_t) dt \right], \quad x \in D,$$

where  $\tau_D$  is the first exit time from  $D$ .

The following estimates for  $\lambda$ -subaveraging functions are general results. Although here they will be used in proving bounds on eigenfunctions and, in consequence, intrinsic ultracontractivity, they may be also of independent interest.

**Theorem 2.1.** *Let Assumptions 2.1–2.4 hold. If  $\varphi$  is a bounded  $\lambda$ -subaveraging function for the semigroup  $\{T_t : t \geq 0\}$  with  $\lambda \geq 0$ , then there is a constant  $C_4 = C_4(X, V, \lambda)$  and  $R = R(X, V, \lambda) > 0$  such that*

$$\varphi(x) \leq C_4 \|\varphi\|_\infty \nu(x), \quad |x| \geq R.$$

The proof of this theorem is probably the most technical and crucial part of the paper. The required bound is obtained inductively, by using a self-improving estimate.

For simplicity, we write  $\mathbf{1}(x)$  instead of  $\mathbf{1}_{\mathbf{R}^d}(x)$  throughout below.

**Theorem 2.2.** *Let Assumptions 2.1–2.4 hold.*

- (1) *If  $\varphi$  is a bounded function (possibly negative) for which there exists  $\lambda > 0$  such that for every  $t > 0$  and  $x \in \mathbf{R}^d$  we have  $e^{\lambda t} T_t \varphi(x) = \varphi(x)$  (clearly, in this case  $|\varphi|$  is  $\lambda$ -subaveraging), then there is a constant  $C_5 = C_5(X, V, \lambda)$  and  $R = R(X, V, \lambda) > 0$  such that*

$$|\varphi(x)| \leq C_5 \|\varphi\|_\infty G_{B(x,1)}^V \mathbf{1}(x) \nu(x), \quad |x| \geq R.$$

- (2) *If  $\varphi$  is a strictly positive function for which there is  $\lambda > 0$  such that for every  $t > 0$  and  $x \in \mathbf{R}^d$  we have  $e^{\lambda t} T_t \varphi(x) = \varphi(x)$ , then there is a constant  $C_6 = C_6(X, \varphi)$  and  $R = R(X, V, \lambda) > 0$  such that*

$$\varphi(x) \geq C_6 G_{B(x,1)}^V \mathbf{1}(x) \nu(x), \quad |x| \geq R.$$

### 2.3. Eigenfunction estimates

The following pointwise upper bounds for eigenfunctions and sharp two-sided bounds for the ground state of the operators  $T_t$  are the next main results of this paper.

**Theorem 2.3 (Upper bounds on eigenfunctions).** *If Assumptions 2.1–2.4 hold, then for every  $\eta \geq 0$  such that  $\lambda_0 + \eta > 0$  there exist constants  $C_7 = C_7(X, V, \eta)$  and a radius  $R = R(X, V, \eta) > 0$  such that*

$$|\varphi_n(x)| \leq C_7 \|\varphi_n\|_\infty G_{B(x,1)}^{V+\eta} \mathbf{1}(x) \nu(x), \quad |x| \geq R, \quad n = 0, 1, 2, \dots$$

**Theorem 2.4 (Ground state estimates).** *If Assumptions 2.1–2.4 hold, then for every  $\eta \geq 0$  such that  $\lambda_0 + \eta > 0$  there exist constants  $C_8 = C_8(X, V, \eta)$ ,  $C_9 = C_9(X, V, \eta)$  and a radius  $R = R(X, V, \eta) > 0$  such that*

$$C_8 G_{B(x,1)}^{V+\eta} \mathbf{1}(x) \nu(x) \leq \varphi_0(x) \leq C_9 G_{B(x,1)}^{V+\eta} \mathbf{1}(x) \nu(x), \quad |x| \geq R.$$

We emphasize that the above bounds on the eigenfunctions are obtained, unlike in [57], without using any (intrinsic) ultracontractivity properties of  $\{T_t : t \geq 0\}$ , which will be further discussed below.

The following surprising domination property (see discussion in Example 2.3 (5) below) is an immediate consequence of the above theorems.

**Corollary 2.1.** *If Assumptions 2.1–2.4 hold, then there is a constant  $C_{10} = C_{10}(X, V)$  such that*

$$|\varphi_n(x)| \leq C_{10} \|\varphi_n\|_\infty \varphi_0(x), \quad x \in \mathbf{R}^d.$$

By the estimates in (3.2) we also have the following corollary.

**Corollary 2.2.** *Let Assumptions 2.1–2.4 hold. Then there exist a constant  $C_{11} = C_{11}(X, V, \lambda)$  and a radius  $R > 0$  such that for all  $|x| \geq R$  we have*

$$|\varphi_n(x)| \leq C_7 \|\varphi_n\|_\infty \frac{\nu(x)}{\inf_{y \in B(x,1)} V(y)}, \quad n = 0, 1, 2, \dots,$$

and

$$C_{11} \frac{\nu(x)}{\sup_{y \in B(x,1)} V(y)} \leq \varphi_0(x) \leq C_9 \frac{\nu(x)}{\inf_{y \in B(x,1)} V(y)}.$$

Moreover, if there is a constant  $C > 1$  such that for all unit balls  $B \subset B(0, R)^c$  it holds that  $\sup_{y \in B} V(y) \leq C \inf_{y \in B} V(y)$  (cf. Assumption 2.5 below), then

$$|\varphi_n(x)| \leq C_7 C \|\varphi_n\|_\infty \frac{\nu(x)}{V(x)}, \quad n = 0, 1, 2, \dots,$$

and

$$C_{11} C^{-1} \frac{\nu(x)}{V(x)} \leq \varphi_0(x) \leq C_9 C \frac{\nu(x)}{V(x)}$$

for all  $|x| \geq R$ .

**Example 2.3.** Some explicit examples of ground state decays are as follows.

- (1) *Rotationally symmetric stable and related processes.* In the case of processes with Lévy intensities satisfying  $\nu(x) \asymp |x|^{-d-\alpha}$ ,  $|x| > 1/2$ ,  $\alpha \in [0, 2)$  (see some listed in Example 2.1 (1)), we have

$$\varphi_0(x) \asymp G_{B(x,1)}^V \mathbf{1}(x) |x|^{-d-\alpha}, \quad |x| > R.$$

Furthermore, when also the condition in Corollary 2.2 is satisfied, then

$$\varphi_0(x) \asymp \frac{1}{(1 + |x|)^{d+\alpha}(1 + V_+(x))}, \quad x \in \mathbf{R}^d.$$

This clearly recovers the results of [46, 47].

- (2) *Symmetric jump Lévy processes with exponentially localized Lévy measure.* It is a well-known result [18, Proposition IV.4] that if  $e^{-t\psi(\cdot)} \in L^1(\mathbf{R}^d)$ ,  $t > 0$ , and there is  $b > 0$  such that  $\int_{\mathbf{R}^d} e^{b|x|} \nu(dx) < \infty$ , then  $|\varphi_n(x)| \leq C e^{-C'|x|}$ ,  $x \in \mathbf{R}^d$ , with  $C = C(X, V, n)$ ,  $C' = C'(X, V, n)$ , i.e., if the Lévy measure is exponentially localized, then the fall-off of the corresponding eigenfunctions is also exponential. Note that Theorems 2.3–2.4 (also Corollary 2.2) essentially improve this result under assumptions which are not much more restrictive than those of [18].
- (3) *Rotationally symmetric relativistic stable process.* This is a special case of the class discussed in (2) above. It was proven in [57, Theorem 1.6] that if  $V$  is a non-negative, locally bounded potential comparable to a rotationally symmetric function, radially non-decreasing and comparable on unit balls with  $\lim_{|x| \rightarrow \infty} V(x)/|x| = \infty$  (i.e., the corresponding Feynman-Kac semigroup is intrinsically ultracontractive), then

$$\varphi_0(x) \asymp \frac{e^{-m^{1/\alpha}|x|}}{(1 + |x|)^{(d+\alpha+1)/2}(1 + V(x))}, \quad x \in \mathbf{R}^d.$$

Theorem 2.4 above generalizes this result to the substantially larger space of  $X$ -Kato class potentials with no restrictions on the order of growth of the potential at infinity and with no use being made of intrinsic ultracontractivity properties. We obtain the result of [57]

as the second part of Corollary 2.2. Note also that Theorem 2.3 is completely new in this context.

- (4) *Symmetric Lévy process with non-degenerate Brownian part.* Let  $(X_t)_{t \geq 0}$  be a process as in Example 2.2 (2). In this case Theorems 2.3 and 2.4 (also Corollary 2.2) allow to identify the leading order of decay of the ground state at infinity (as well as provide upper bounds for higher order eigenfunctions) as the contribution of the Lévy intensity  $\nu$  and a correction from the potential  $V$ . However, since our constants are not optimal, it can be expected that the correct asymptotics should contain a further term of smaller order (similar to Carmona's bound in [16]) coming from the Brownian component of the process. However, showing this requires a more subtle argument and cannot be seen from our present results.

- (5) *Diffusions.* It is useful to compare our results to the classic facts known for the eigenfunctions of Feynman-Kac semigroups involving Brownian motion (i.e., Schrödinger semigroups generated by  $-\Delta + V$ ). In the general case,  $t_0$ -GSD and  $t_0$ -IUC (see the definitions in Subsection 2.4) imply directly that for all  $x \in \mathbf{R}^d$

$$(2.4) \quad |\varphi_n(x)| \leq C(t_0) \|\varphi_n\|_\infty e^{(\lambda_n - \lambda_0)t_0} \varphi_0(x) \quad \text{and} \quad |\varphi_n(x)| \leq C(t_0) e^{(\lambda_n - \lambda_0)t_0} \varphi_0(x),$$

respectively, which is similar to the domination property in Corollary 2.1. However, for diffusions this property cannot be taken for granted in lack of AGSD/AIUC. This can be seen, for instance, in the example of the harmonic oscillator, for which  $V(x) = |x|^2$  and the eigenfunctions are given by the Hermite functions. In this case the semigroup is not IUC (not even AIUC), and the property in Corollary 2.1 does not occur either. This surprisingly contrasts the case of jump processes we consider, for which a version of (2.4) occurs without being AGSD/AIUC.

## 2.4. Intrinsic ultracontractivity and ground state domination

Under the given choice of potential the Feynman-Kac semigroup has strong smoothing properties which we define next. In particular, they imply degrees of regularity and the rate of decay of eigenfunctions. Recall that the intrinsic fractional Feynman-Kac semigroup is given by (1.1).

**Definition 2.2 (IUC/AIUC).** *Consider the following ultracontractivity properties.*

- (1) *The semigroup  $\{T_t : t \geq 0\}$  is ultracontractive if  $T_t$  is a bounded operator from  $L^2(\mathbf{R}^d)$  to  $L^\infty(\mathbf{R}^d)$ , for every  $t > 0$ .*
- (2) *The semigroup  $\{T_t : t \geq 0\}$  is intrinsically ultracontractive (abbreviated as IUC) if  $\tilde{T}_t$  is a bounded operator from  $L^2(\mathbf{R}^d, \varphi_0^2 dx)$  to  $L^\infty(\mathbf{R}^d)$ , for every  $t > 0$ .*
- (3) *The semigroup  $\{T_t : t \geq 0\}$  is  $t_0$ -intrinsically ultracontractive (abbreviated as  $t_0$ -IUC) if the above boundedness property of  $\tilde{T}_t$  holds for some  $t_0 > 0$ . In this case the semigroup property extends it to all  $t \geq t_0$ .*
- (4) *When  $\tilde{T}_t$  is  $t_0$ -IUC, but the specific value of  $t_0$  is not essential, we simply say that  $\{T_t : t \geq 0\}$  is asymptotically intrinsically ultracontractive (abbreviated as AIUC).*

Since below we mainly use probabilistic arguments, it will be useful to consider the following property of the semigroup  $\{T_t : t \geq 0\}$  which we call *ground state domination*. As it will be seen later, in general ground state domination is a weaker property than IUC.

**Definition 2.3 (GSD/AGSD).** *Consider the following boundedness properties.*

- (1) The semigroup  $\{T_t : t \geq 0\}$  is ground state dominated (abbreviated as GSD) if for every  $t > 0$  there is a constant  $C_{12} = C_{12}(X, V, t)$  such that

$$(2.5) \quad T_t \mathbf{1}(x) \leq C_{12} \varphi_0(x), \quad x \in \mathbf{R}^d.$$

- (2) The semigroup  $\{T_t : t \geq 0\}$  is  $t_0$ -ground state dominated (abbreviated as  $t_0$ -GSD) if (2.5) holds for some  $t_0 > 0$ . In this case the semigroup property extends this bound to all  $t \geq t_0$  with constant  $C_{12} e^{-\lambda_0(t-t_0)}$ , where  $C_{12} = C_{12}(X, V, t_0)$ .
- (3) When  $\tilde{T}_t$  is  $t_0$ -GSD but the specific value of  $t_0$  is not essential, we simply say that  $\{T_t : t \geq 0\}$  is asymptotically ground state dominated (abbreviated as AGSD).

Before stating the main results of this subsection it is worthwhile to discuss the relationship between these properties. Parts (1)-(2) in Definition 2.2 are standard, (4) has been introduced in [47]. It is immediate from the definitions that IUC and GSD imply  $t_0$ -IUC and  $t_0$ -GSD (for all  $t_0 > 0$ ), respectively. However, as it will be seen below, IUC and GSD are essentially stronger properties than their asymptotic versions. We will show that under our assumptions AIUC and AGSD are equivalent, while IUC implies GSD and in general conversely this is not the case.

**Theorem 2.5 ((A)IUC and (A)GSD).** *Let Assumptions 2.1–2.4 be satisfied, specifically, let Assumption 2.2 hold with  $t_b > 0$ .*

- (1) Then

$$AIUC \iff AGSD \quad \text{and} \quad IUC \implies GSD$$

*in the sense that  $t_0$ -IUC  $\implies 2t_0$ -GSD,  $t_0 > 0$ , and  $t_0$ -GSD  $\implies 2t_0$ -IUC, whenever  $t_0 \geq t_b$ .*

- (2) If, moreover,  $p(t, x) \leq C_4$  for all  $t > 0$  and  $x \in \mathbf{R}^d$  with  $C_4 = C_4(X, t)$ , then also

$$GSD \implies IUC.$$

In Proposition 2.1 below we show that the assumption in part (2) of Theorem 2.5 is essential. This means that in general when  $p(t, \cdot)$  may be unbounded for small  $t$ , IUC is a strictly stronger property than GSD. Intuitively it is clear that IUC requires more smoothness of the semigroup  $\{T_t : t \geq 0\}$  than GSD as it also depends on the local singularities of the semigroup, while GSD is in fact, roughly speaking, a decay property of the semigroup at infinity.

We now present characterization results on GSD/AGSD and IUC/AIUC.

**Theorem 2.6 (Sufficient and necessary conditions for AGSD).** *Let Assumptions 2.1–2.4 hold.*

- (1) If there exist a constant  $C_{13} = C_{13}(X, V)$  and a radius  $R > 0$  such that

$$\frac{V(x)}{|\log \nu(x)|} \geq C_{13}, \quad |x| \geq R,$$

*then the semigroup  $\{T_t : t \geq 0\}$  is  $t_0$ -GSD with  $t_0 = 4/C_{13}$ .*

- (2) If the semigroup  $\{T_t : t \geq 0\}$  is  $t_0$ -GSD, then for every  $\varepsilon \in (0, 1]$  there is  $R_\varepsilon > 0$  such that

$$\frac{\sup_{y \in B(x, \varepsilon)} V(y)}{|\log \nu(x)|} \geq \frac{1}{2t_0}, \quad |x| \geq R.$$

**Theorem 2.7 (Sufficient and necessary conditions for GSD).** *Let Assumptions 2.1–2.4 hold.*

- (1) If

$$\lim_{|x| \rightarrow \infty} \frac{V(x)}{|\log \nu(x)|} = \infty,$$

*then the semigroup  $\{T_t : t \geq 0\}$  is GSD.*

(2) If the semigroup  $\{T_t : t \geq 0\}$  is GSD, then for every  $\varepsilon \in (0, 1]$  we have

$$\lim_{|x| \rightarrow \infty} \frac{\sup_{y \in B(x, \varepsilon)} V(y)}{|\log \nu(x)|} = \infty,$$

**Remark 2.1.** By Theorem 2.5 the limit condition in Theorem 2.7 (2) is necessary for IUC, and the condition in Theorem 2.7 (1) is sufficient for IUC, whenever  $p(t, \cdot)$  is bounded for all  $t > 0$ . Similarly, the growth condition in Theorem 2.6 (2) is necessary for  $t_0/2$ -IUC, and the condition in Theorem 2.6 (1) is sufficient for  $t_0$ -IUC with  $t_0 = 2(t_b \vee 4/C_{13})$ .

The following result is intuitively clear, however, for the reader's convenience we include a short proof at the end of Subsection 3.1.

**Proposition 2.1 (Ultracontractivity).** *Let Assumptions 2.1–2.4 hold, in particular, let Assumption 2.2 hold with  $t_b > 0$ . Furthermore, suppose that*

- (1) *there exists  $t < t_b$  such that  $\lim_{|x| \rightarrow 0^+} p(t, x) = \infty$  and for all  $s \in (0, t]$ ,  $p(s, x) \geq p(s, y)$  whenever  $|x| \leq |y|$*
- (2) *there exist  $x_0 \in \mathbf{R}^d$  and  $\varepsilon > 0$  such that  $V$  is bounded from above in  $B(x_0, \varepsilon)$ .*

*Then for every  $0 < t < t_b$  for which condition (1) is satisfied, the operator  $T_{t/2}$  is not bounded from  $L^2(\mathbf{R}^d)$  to  $L^\infty(\mathbf{R}^d)$ . In particular, the semigroup  $\{T_t : t \geq 0\}$  is not ultracontractive.*

Since  $\varphi_0$  is bounded, IUC implies ultracontractivity. Hence the above result shows that the assumption in assertion (2) of Theorem 2.5 is essential. This means that there exists a class of random processes whose Feynman-Kac semigroups are GSD but not IUC (even if the potential grows to infinity at infinity very quickly). Typical examples of Lévy processes fitting the above proposition include subordinate Brownian motion with suitably slowly varying characteristic exponents such as geometric stable processes. This example will be discussed in more detail in the next subsection.

For the remainder of this subsection we restrict attention to a somewhat smaller class of potentials by imposing more regularity. This will also be used in the next subsection.

**Assumption 2.5.** *There exist  $R > 1$  and a constant  $C_{14} = C_{14}(V)$  such that for every  $|x| > R$*

$$(2.6) \quad V(y) \leq C_{14} V(x), \quad y \in B(x, 1)$$

*holds.*

A straightforward consequence of the above theorems is the following result.

**Corollary 2.3 (Borderline case).** *Let Assumptions 2.1–2.5 hold, in particular, let Assumption 2.2 hold with  $t_b > 0$ . Then we have the following.*

- (1) *The semigroup  $\{T_t : t \geq 0\}$  is GSD if and only if*

$$(2.7) \quad \lim_{|x| \rightarrow \infty} \frac{V(x)}{|\log \nu(x)|} = \infty.$$

*Moreover, condition (2.7) is necessary for IUC, and sufficient whenever  $p(t, \cdot)$  is bounded for every fixed  $t > 0$ .*

- (2) *The semigroup  $\{T_t : t \geq 0\}$  is AGSD (or, equivalently, AIUC) if and only if there exist a constant  $C_{15}$  and  $R > 0$  such that*

$$(2.8) \quad \frac{V(x)}{|\log \nu(x)|} \geq C_{15}, \quad |x| \geq R.$$

Specifically, if (2.8) is satisfied, then  $\{T_t : t \geq 0\}$  always is  $t_0$ -GSD with  $t_0 = 4/C_{15}$ , and it is  $t_0$ -IUC with  $t_0 = 2(t_b \vee 4/C_{15})$ . If  $\{T_t : t \geq 0\}$  is  $t_0$ -GSD, then (2.8) holds with constant  $C_{15} = 1/(2C_{14}t_0)$ . Similarly,  $t_0$ -IUC implies (2.8) with  $C_{15} = 1/(4C_{14}t_0)$ .

By the above results we are now able to formally define borderline potentials.

**Definition 2.4 (Borderline potential).** *Let Assumptions 2.1–2.4 hold. We call  $V$  borderline potential for  $(A)$ GSD/ $(A)$ IUC of the semigroup  $\{T_t : t \geq 0\}$  if there exist  $t_0 > 0$  and  $R > 0$  such that  $t_0 V(x) = |\log \nu(x)|$ , for every  $x \in B(0, R)^c$ .*

Note that by Assumption 2.1 the borderline potentials always satisfy Assumption 2.5. Also, note that we speak of borderline potentials in the sense of equivalence classes given by the definition above.

## 2.5. Probabilistic and variational interpretation of AGSD/AIUC with examples

It was seen in the previous subsection that under Assumptions 2.1–2.4 AGSD/AIUC of  $\{T_t : t \geq 0\}$  depends only on the intensity of large jumps of the process  $(X_t)_{t \geq 0}$ . This means that whenever  $\nu \neq 0$ , the Gaussian and small jump parts in the Lévy-Itô decomposition of  $(X_t)_{t \geq 0}$  have no impact on AGSD/AIUC. Indeed, the borderline growth of  $V$  is decided by the ratio  $e^{-t_0 V(x)}/\nu(x)$  for  $x$  sufficiently far away from the origin and some time point  $t_0 > 0$ . More precisely, AGSD/AIUC occurs if and only if  $e^{-t_0 V(x)}$  is uniformly dominated by the jump intensity  $\nu(x)$  outside a bounded set in  $\mathbf{R}^d$ . We note that although this description gives a full picture of what AGSD/AIUC is in the case when  $\nu \neq 0$ , it does not help to understand what is behind this property when the process is strictly diffusive, i.e., whenever  $\nu = 0$ . (In a sense, this situation confirms that Brownian motion is an exceptional Lévy process and processes with jumps are the more generic.) In this section we discuss probabilistic and variational descriptions of these properties.

It is straightforward that the condition on  $V$  for  $\{T_t : t \geq 0\}$  being AGSD/AIUC is much weaker than in the case of the Feynman-Kac semigroup for diffusions. This can be explained by the following heuristic interpretation. For our purposes here it suffices to observe that the effect of the potential on the distribution of paths is a concurrence of killing at a rate of  $e^{-\int_0^t V_+(X_s)ds}$  and mass generation at a rate of  $e^{\int_0^t V_-(X_s)ds}$ . When, however,  $V(x) \rightarrow \infty$  as  $|x| \rightarrow \infty$ , then outside a compact set only the killing effect occurs and  $\mathbf{E}^x[e^{-\int_0^t V(X_s)ds}]$  gives the probability of survival of the process up to time  $t$ . The following characterization of AGSD/AIUC may be used as a probabilistic definition of these properties, valid for both our jump Lévy processes and Brownian motion. In fact, this property has a strong ergodic flavour, compare also with [33, 53].

**Proposition 2.2.** *The semigroup  $\{T_t : t \geq 0\}$  is AGSD/AIUC if and only if there exist  $t > 0$ , a bounded non-empty Borel set  $D \subset \mathbf{R}^d$ , and a constant  $C_{16} = C_{16}(X, V, t)$  such that for every Borel set  $A \subseteq \mathbf{R}^d$  we have*

$$(2.9) \quad \mathbf{E}^x \left[ e^{-\int_0^t V(X_s)ds}; X_t \in A \right] \leq C_{16} \mathbf{E}^x \left[ e^{-\int_0^t V(X_s)ds}; X_t \in D \right], \quad x \in \mathbf{R}^d.$$

(For a proof see [47, Cor. 4.1, Prop. 4.1] and [57, eqs. (1.2)–(1.3)].) Asymptotically, the probability of survival of the process “staying around the starting point  $x$ ” (far from the region  $D$ ) is approximately  $e^{-tV(x)}$ , while the probability of surviving by escaping to a region  $D$  with a lower killing rate is  $\mathbf{P}^x(X_t \in D)$ . By using (2.9) it is immediately seen that when  $\{T_t : t \geq 0\}$  is AGSD/AIUC, then the probability that the process under  $V$  survives up to time  $t$  far from the location of  $\inf V$  is bounded by the probability that the process survives up to time  $t$  and is in some bounded region

$D$ , no matter its starting point. It can be expected that the balance of the competing effects in fact will be decided roughly by the ratio  $V(x)/|\log \mathbf{P}^x(X_t \in D)|$ . Below we prove this intuition and show that for a large class of non-diffusive Lévy processes the expression  $|\log \mathbf{P}^x(X_t \in D)|$  precisely determines the borderline potential. Note that this expression does not give the borderline potential for diffusions, however, it allows to identify the leading order of the borderline growth which is known to be quadratic [32]. Some further examples will be discussed below.

The following comparability condition will be used in Propositions 2.3–2.4 only. It is partly satisfied under our previous assumptions and, in general, it seems to be strongly related to Assumption 2.1. However, we don't know any general argument that can be used to show possible implication. Therefore, and also for more clarity, we decided not to discuss this condition and formulate it as an independent assumption.

**Assumption 2.6.** *For every  $t > 0$  there is  $R = R(t) > 0$  such that*

$$|\log \nu(x)| \asymp C_{17} |\log p(t, x)| \asymp C_{18} |\log \mathbf{P}^x(X_t \in B(0, 1))|, \quad |x| > R,$$

with constants  $C_{17} = C_{17}(X)$  and  $C_{18} = C_{18}(X)$  (independent of  $t$ ).

The next two propositions are direct consequences of Assumption 2.6 and Theorems 2.6–2.7.

**Proposition 2.3 (AGSD/AIUC probabilistically).** *Let Assumptions 2.1–2.4 and 2.6 be satisfied. Then the following hold:*

- (1) *If the semigroup  $\{T_t : t \geq 0\}$  is  $t_0$ -GSD (or  $t_0/2$ -IUC), then for every  $0 < \varepsilon \leq 1$  there is  $R \geq 2$  such that*

$$\frac{\sup_{y \in B(x, \varepsilon)} V(y)}{|\log \mathbf{P}^x(X_{t_0} \in B(0, 1))|} \geq \frac{1}{2C_{17}^2 C_{18} t_0} \quad \text{and} \quad \frac{\sup_{y \in B(x, \varepsilon)} V(y)}{|\log p(t_0, x)|} \geq \frac{1}{2C_{17} t_0}, \quad |x| \geq R.$$

*Moreover, if also Assumption 2.5 holds, then  $\sup_{y \in B(x, \varepsilon)} V(y)$  can be replaced by  $C_V V(x)$ .*

- (2) *If there exist  $t > 0$ ,  $R > 0$  and a constant  $C_{19} = C_{19}(X, V)$  such that*

$$\frac{V(x)}{|\log \mathbf{P}^x(X_t \in B(0, 1))|} \geq \frac{1}{C_{19} t} \quad \text{or} \quad \frac{V(x)}{|\log p(t, x)|} \geq \frac{1}{C_{19} t}, \quad |x| > R,$$

*then  $\{T_t : t \geq 0\}$  is  $t_0$ -GSD with  $t_0 = 4C_{18}C_{19}t$  and  $t_0$ -IUC with  $t_0 = 2(t_b \vee 4C_{18}C_{19}t)$  or  $t_0$ -GSD with  $t_0 = 4C_{17}C_{19}t$  and  $t_0$ -IUC with  $t_0 = 2(t_b \vee 4C_{17}C_{19}t)$ , respectively.*

**Proposition 2.4 (GSD/IUC probabilistically).** *Let Assumptions 2.1–2.4 and 2.6 be satisfied. Then the following hold:*

- (1) *If the semigroup  $\{T_t : t \geq 0\}$  is GSD (or IUC), then for every  $t > 0$  we have*

$$\lim_{|x| \rightarrow \infty} \frac{\sup_{y \in B(x, \varepsilon)} V(y)}{|\log \mathbf{P}^x(X_t \in B(0, 1))|} = \lim_{|x| \rightarrow \infty} \frac{\sup_{y \in B(x, \varepsilon)} V(y)}{|\log p(t, x)|} = \infty.$$

*When in addition also Assumption 2.5 holds, then  $\sup_{y \in B(x, \varepsilon)} V(y)$  may be replaced by  $V(x)$ .*

- (2) *If there is  $t > 0$  such that*

$$\lim_{|x| \rightarrow \infty} \frac{V(x)}{|\log \mathbf{P}^x(X_t \in B(0, 1))|} = \infty \quad \text{or} \quad \lim_{|x| \rightarrow \infty} \frac{V(x)}{|\log p(t, x)|} = \infty,$$

*then  $\{T_t : t \geq 0\}$  is GSD. If, moreover,  $p(t, \cdot)$  is bounded for all  $t > 0$ , then any of these two conditions also implies IUC.*

We now illustrate these results by specific examples.



**Example 2.4.** Theorems 2.6–2.7 and Propositions 2.3–2.4 apply directly to the following three classes of examples with different growth rate of borderline potentials. For any of these specific examples, Assumption 2.6 can be verified by using the time-space estimates of the related transition densities, similarly as in Example 2.2.

- (1) *Borderline potentials of logarithmic order.* The borderline behaviour

$$-\log \nu(x) \asymp -\log \mathbf{P}^x(X_t \in B(0, 1)) \asymp -\log p(t, x) \asymp \log |x|$$

occurs in the following cases (see Example 2.3 (1)):

- (i) *Purely jump stable-type processes with bounded transition densities:* This includes (a) non-Gaussian rotationally symmetric stable processes (our results recover and substantially improve the methods of [46, 47]), (b) mixtures of rotationally symmetric stable processes, (c) rotationally symmetric stable-like Lévy processes. In this case Theorem 2.5 implies equivalence of GSD and IUC.
- (ii) *Rotationally symmetric geometric  $\alpha$ -stable processes ( $\alpha \in (0, 2)$ ):* By Proposition 2.1 the semigroup  $\{T_t : t \geq 0\}$  is not IUC (i.e., it is not  $t$ -IUC for small  $t > 0$ , not even ultracontractive), while it is GSD and  $t$ -IUC for  $t > 2t_b$  provided the potential  $V$  is pinning enough.
- (iii) *Jump-diffusions* (see Example 2.2(1)(iii)): Since  $\nu \neq 0$ , the Gaussian component has no effect on (A)GSD and (A)IUC.

- (2) *Borderline potentials of linear order.* The borderline behaviour

$$-\log \nu(x) \asymp -\log \mathbf{P}^x(X_t \in B(0, 1)) \asymp -\log p(t, x) \asymp |x|$$

occurs for rotationally symmetric Lévy processes satisfying our assumptions provided that their Lévy intensities are exponentially decaying at infinity. Important examples to this class are rotationally symmetric relativistic stable processes and tempered rotationally symmetric stable processes (with  $\nu(x) \asymp e^{-a|x|}|x|^{-d-\alpha}$ ,  $a > 0$ ,  $\alpha \in (0, 2)$ , compare Example 2.3 (2)-(3)). For relativistic stable processes it was proven in [57] that when  $V$  is a non-negative, locally bounded potential comparable to a function which is rotationally symmetric, radially non-decreasing and comparable on unit balls, then the corresponding Feynman-Kac semigroup is IUC if and only if  $\lim_{|x| \rightarrow \infty} V(x)/|x| = \infty$ . The combination of Theorems 2.5 and 2.7 generalizes this result to the substantially larger class of  $X$ -Kato class potentials. Also, we obtain the result of [57] in Corollary 2.3 (1) under Assumption 2.5.

- (3) *Borderline potentials of sublinear but faster than logarithmic order.* The borderline behaviour

$$-\log \nu(x) \asymp -\log \mathbf{P}^x(X_t \in B(0, 1)) \asymp -\log p(t, x) \asymp |x|^\beta, \quad 0 < \beta < 1$$

appears in the case of processes with Lévy measures decaying sub-exponentially at infinity, discussed in Example 2.2 (4) (case  $\beta < 1$ ).

Note that, roughly speaking, this is the complete range of possible borderline growths for the processes we consider. An asymptotic growth of the order  $|x|^\beta$  with  $\beta > 1$  is ruled out by Assumption 2.1 (3), see the discussion in Example 2.1 (4). Also, the borderline potential cannot be slower than logarithmic due to the integrability of the Lévy intensity  $\nu$  outside a neighbourhood of the origin. We also note that linear growth is the quickest possible as well for the class of subordinate Brownian motions obtained under subordinators whose Lévy exponents are complete Bernstein functions [50, Lemma 2.1].

A second type of example is about Feynman-Kac (in fact, Schrödinger) semigroups involving standard Brownian motion under a potential. Although the strictly diffusive case when  $\nu \equiv 0$  is not covered by our paper, it is interesting to compare the results to better understand what mechanism lies behind IUC in the general case.

**Example 2.5. Diffusions.** In the classic papers on IUC of Schrödinger semigroups generated by  $-\Delta + V$  it was considered whether the property holds for some special ways of choosing the potential. In the one-dimensional case [32, Theorem 6.1] shows that when  $V(x) = |x|^a$ ,  $a > 0$ , or  $V(x) = |x|^2 \log(|x| + 2)^b$ ,  $b > 0$ , then the related semigroup is IUC if and only if  $a > 2$  and  $b > 2$  (i.e., fails for  $a, b \leq 2$ ). When  $d \geq 1$  it is shown in [32, Theorem 6.3] that IUC occurs whenever  $C_1 + C_2|x|^b \leq V(x) \leq C_3 + C_4|x|^a$ , with  $a/2 + 1 < b$ . To the best of our knowledge, AIUC was not considered before the paper [47]. However, by a use of the Mehler formula it follows that in the case of the harmonic oscillator AIUC does not occur [31, Theorem 4.3.2]. Recently, in [1] a general sufficient condition for IUC was found for Schrödinger semigroups. For radial potentials  $V$  this condition is also necessary and it is formulated as

$$(2.10) \quad \int_{r_0}^{\infty} \frac{1}{\sqrt{V(r)}} dr < \infty, \quad \text{for some } r_0 > 0.$$

For instance, for the potential

$$V(x) = |x|^2 (\log |x|)^2 (\log \log |x|)^2 \dots \underbrace{(\log \dots \log |x|)^2}_{(m-1)\text{-times}} \underbrace{(\log \dots \log |x|)^{2+\delta}}_{m\text{-times}}, \quad m \in \mathbf{N}, \delta \geq 0,$$

this condition is satisfied if and only if  $\delta > 0$ . This means that IUC holds for an arbitrary choice of  $m \in \mathbf{N}$  whenever  $\delta > 0$ , and suggests that in the diffusion case it is not possible to identify the borderline potential directly as in the case of jump processes. Note that all of the classic results discussed above were obtained by purely analytic arguments. We believe that it is possible to derive the analytic condition (2.10) by probabilistic methods based on sufficiently efficient estimates of the expression at the right hand side of (2.9). For instance, when  $V \geq 0$  satisfies Assumption 2.5 and the semigroup is IUC, then a rough estimate gives  $\lim_{|x| \rightarrow \infty} V(x) / |\log \mathbf{P}^x(X_t \in D)| = \infty$ , allowing correctly to identify  $|x|^2$  as the leading order of borderline growth, as in Proposition 2.4(1).

In the context of diffusions, we also mention that a condition similar to part (3) of Assumption 2.1 has been used for Green functions of elliptic differential operators on domains in [65, 63, 77] and related papers, and it goes back to [66, 67], where it was introduced as a small-perturbation condition of an elliptic operator by another operator. In particular, it is shown that intrinsic ultracontractivity implies the small-perturbation condition.

Finally, we give another description of AGSD/AIUC. In order to do that we need to put one more condition on the Lévy measure.

**Assumption 2.7.** For every  $R > 0$  we have  $\log \nu \in L^1(B(0, R)^c, \nu(x)dx)$ .

Under Assumptions 2.1–2.5 and 2.7, and for all  $A \in \mathcal{B}(\mathbf{R}^d)$  such that  $\text{dist}(A, 0) > 0$  we define the functionals

$$(2.11) \quad E_A^V(\nu) = \int_A V(x) \nu(x) dx$$

$$(2.12) \quad H_A(\nu) = - \int_A \nu(x) \log \nu(x) dx$$

$$(2.13) \quad F_A^V(\nu) = E_A^V(\nu) - H_A(\nu).$$

Note that under Assumption 2.7  $F_A^V(\nu)$  is well defined. We call the functional  $E_A^V(\nu)$  *energy*,  $H_A(\nu)$  *entropy*, and  $F_A^V(\nu)$  *free energy* in set  $A$  for the given potential  $V$  and Lévy measure  $\nu(dx)$ . Note that since  $\nu(x)$  is the Radon-Nikodým derivative of the Lévy measure with respect to Lebesgue measure,  $H_A(\nu)$  is in fact the relative entropy (or Kullback-Leibler functional) of the Lévy measure with respect to Lebesgue measure. Then we have the following characterization of AGSD/AIUC.

**Theorem 2.8 (Characterization of AGSD/AIUC).** *Let Assumptions 2.1–2.5 and 2.7 hold. The potential  $V$  is such that the semigroup  $\{T_t : t \geq 0\}$  is AGSD (or, equivalently, AIUC) if and only if there exists  $t_0 > 0$  and  $R > 0$  such that for every Borel set  $A \subset B(0, R)^c$  we have that  $F_A^{t_0 V}(\nu) \geq 0$ . Specifically, if  $\{T_t : t \geq 0\}$  is  $t_0$ -GSD, then  $F_A^{2C_{14}t_0 V}(\nu) \geq 0$ . If  $F_A^{t_0 V}(\nu) \geq 0$ , then  $\{T_t : t \geq 0\}$  is  $8C_{14}t_0$ -GSD.*

Note that due to monotonicity of the free energy functional with respect to potential  $V$ , the inequality  $F_A^{t_0 V}(\nu) \geq 0$  implies  $F_A^{tV}(\nu) \geq 0$  for all  $t \geq t_0$ . Furthermore, we have the following variational result.

**Corollary 2.4 (Variational principle for borderline potential).** *Let Assumptions 2.1–2.4 and 2.7 hold, and the jump intensity  $\nu$  and the potential  $V$  be continuous functions. Then  $V$  is the borderline potential for AGSD/AIUC of the semigroup  $\{T_t : t \geq 0\}$  if and only if there exist  $t_0 > 0$  and  $R > 0$  such that  $F_A^{t_0 V}(\nu) = 0$  for all Borel sets  $A \subset B(0, R)^c$ .*

We note that similar energy and entropy functionals have been used in [36] to determine heavy tailed probability distributions with prescribed asymptotics, satisfying the Fokker-Planck equation. Such optimization methods are widely used, however, in our context it is derived and rigorously justified by Theorem 2.8. Furthermore, the above variational problem can also be considered in the reverse direction. Roughly speaking, for a given sufficiently regular potential  $V$  we may be interested in finding the appropriate Lévy measures  $\nu$  (i.e., Lévy processes  $(X_t)_{t \geq 0}$ ) such that the corresponding free energy functional  $F_A^{t_0 V}(\nu)$  is minimized for some  $t_0 > 0$ ,  $R > 0$  and every Borel set  $A \subset B(0, R)^c$ .

### 3. Proofs

#### 3.1. Preliminaries

Here we recall some basic facts of potential theory for the Feynman-Kac semigroup related to process  $(X_t)_{t \geq 0}$  needed for our purposes, and show some technical facts used in proving our results concerning intrinsic ultracontractivity and the eigenfunction estimates below. For background we refer to [30, 11, 12, 26, 27, 10].

We adopt the convention that auxiliary constants appearing in proofs may change their values from one use to another (possibly from line to line). However, if necessary, we write  $C, C^{(1)}, C^{(2)}, \dots$  to distinguish them. Recall that, in contrast, constants appearing in the statements of theorems, propositions and lemmas are fixed throughout the paper and can be tracked in the proofs.

Denote by

$$e_V(t) := e_V(t)(\omega) = e^{-\int_0^t V(X_s(\omega)) ds}, \quad t > 0.$$

the Feynman-Kac-functional for the Lévy process  $(X_t)_{t \geq 0}$  for potential  $V$ . By standard arguments based on the Khasminskii's lemma [30, Cor.Prop. 3.8] there are constants  $C_{21} = C_{21}(X, V)$  and

$C_{22} = C_{22}(X, V)$  such that

$$(3.1) \quad \sup_{x \in \mathbf{R}^d} \mathbf{E}^x[e_V(t)] \leq \sup_{x \in \mathbf{R}^d} \mathbf{E}^x[e_{-V_-}(t)] \leq C_{21}e^{C_{22}t}, \quad t > 0.$$

Recall that  $\tau_D = \inf\{t > 0 : X_t \notin D\}$  denotes the first exit time of the process from the set  $D$ . The potential operator for the semigroup  $\{T_t : t \geq 0\}$  is defined by

$$G^V f(x) = \int_0^\infty T_t f(x) dt = \mathbf{E}^x \left[ \int_0^\infty e_V(t) f(X_t) dt \right],$$

for non-negative or bounded Borel functions  $f$  on  $\mathbf{R}^d$ , while the  $V$ -Green operator for an open set  $D$  is given by

$$G_D^V f(x) = \int_0^\infty \mathbf{E}^x [t < \tau_D; e_V(t) f(X_t)] dt = \mathbf{E}^x \left[ \int_0^{\tau_D} e_V(t) f(X_t) dt \right],$$

for non-negative or bounded Borel functions  $f$  on  $D$ .

It can be seen directly that if  $D \subset \mathbf{R}^d$  is a non-empty bounded open set and  $V$  is a non-negative and not identically zero potential on  $D$ , then for all  $x \in D$  we have

$$(3.2) \quad \left( 1 - \exp(-\sup_{y \in D} V(y)) \right) \frac{\mathbf{P}^x(\tau_D > 1)}{\sup_{y \in D} V(y)} \leq G_D^V \mathbf{1}(x) \leq \frac{1}{\inf_{y \in D} V(y)}.$$

Here we use the convention that  $1/\infty = 0$  and  $1/0^+ = \infty$ .

Below we often use the fact that for all bounded Borel sets  $D \subset \mathbf{R}^d$  and  $x \in D$  we have  $\mathbf{E}^x[\tau_D] \leq \mathbf{E}^x[\tau_{B(x, \text{diam } D)}] = \mathbf{E}^0[\tau_{B(0, \text{diam } D)}] < \infty$ . Furthermore, when  $D' \subset \mathbf{R}^d$  is an open set,  $D \subset D'$  is open and bounded, and  $f$  is a non-negative or bounded Borel function on  $D'$ , then by the strong Markov property it follows for every  $x \in D$  that

$$(3.3) \quad G_{D'}^V f(x) = G_D^V f(x) + \mathbf{E}^x [X_{\tau_D} \in D' \setminus D; e_V(\tau_D) G_{D'}^V f(X_{\tau_D})].$$

A Borel function  $f$  on  $\mathbf{R}^d$  is called  $(X, V)$ -harmonic in an open set  $D \subset \mathbf{R}^d$  if

$$(3.4) \quad f(x) = \mathbf{E}^x [\tau_U < \infty; e_V(\tau_U) f(X_{\tau_U})], \quad x \in U,$$

for every open set  $U$  with  $\overline{U}$  contained in  $D$ , and it is called regular  $(X, V)$ -harmonic in  $D$  if (3.4) holds for  $U = D$ . By the strong Markov property every regular  $(X, V)$ -harmonic function in  $D$  is  $(X, V)$ -harmonic in  $D$ . We always assume that the expectation in (3.4) is absolutely convergent.

The following uniform estimates for local suprema of  $(X, V)$ -harmonic functions are an important ingredient in proving AGSD/GSD and eigenfunction bounds. Under Assumptions 2.1-2.3 they directly follow from the more general results in [14].

**Lemma 3.1.** *Let Assumptions 2.1 (1), 2.2 and 2.3 be satisfied. Then for every  $0 < r < p < q < R \leq 1$  there exists a constant  $C_{23} = C_{23}(X, r, p, q, R)$  such that for any  $V \in \mathcal{K}_{\text{loc}}^X$ ,  $V \geq 0$  on  $B(x_0, R)$ , and every non-negative function  $f$  on  $\mathbf{R}^d$  that is  $(X, V)$ -regular harmonic on  $B(x_0, R)$ , we have*

$$f(y) \asymp C_{23} G_{B(x_0, p)}^V \mathbf{1}(y) \int_{B(x_0, q)^c} f(z) \nu(z - x_0) dz, \quad |y - x_0| < r.$$

*Proof.* Under the assumptions of the lemma Assumptions A-D in [14] are satisfied. Specifically, since  $(X_t)_{t \geq 0}$  is a symmetric strong Feller Lévy process, Assumptions A and B hold directly, while Assumption C is a consequence of our Assumption 2.1 (1), and our Assumption 2.3 is just Assumption D (for details of their verification see [14, Example 5.5]). Thus the above estimates hold for  $V \equiv 0$  as a consequence of [14, Lemma 3.2 and Theorem 3.4] for the set  $D = B(x_0, R)$ . By space homogeneity of  $(X_t)_{t \geq 0}$  the constant  $C_{23}$  does not depend on the specific  $x_0$ . Similar estimates for

an arbitrary  $V \in \mathcal{K}_{\text{loc}}^X$ ,  $V \geq 0$  on  $B(x_0, R)$ , i.e., for the subprocess of  $(X_t)_{t \geq 0}$  given by the multiplicative functional  $M_t = e_V(t)$ , follow from the latter with the same constant  $C_{23}$  (independent of  $V$ ) by the argument in [14, Example 5.9].  $\square$

Note that it is crucial below that  $C_{23}$  in the above bounds does not depend on the (local behaviour of) potential  $V$ . It is also essential for our further applications that due to space-homogeneity of the process  $(X_t)_{t \geq 0}$ , the constant is independent of the location in space of the ball  $B(x_0, R)$ . In fact, Lemma 3.1 will be used below in the following form which is sufficiently general and more convenient for our purposes.

**Corollary 3.1.** *Let Assumptions 2.1 (1), 2.2 and 2.3 be satisfied. Then there exists a constant  $C_{24} = C_{24}(X)$  such that for any  $V \in \mathcal{K}_{\text{loc}}^X$ ,  $V \geq 0$  on  $B(x_0, 1)$ , and every non-negative function  $f$  on  $\mathbf{R}^d$  that is  $(X, V)$ -regular harmonic on  $B(x_0, 1)$ , we have*

$$f(y) \asymp C_{24} G_{B(x_0, 1)}^V \mathbf{1}(y) \int_{B(x_0, 3/4)^c} f(z) \nu(z - x_0) dz, \quad |y - x_0| < \frac{1}{2}.$$

*Proof.* By taking  $r = 1/2$ ,  $p = 5/8$ ,  $q = 3/4$  and  $R = 1$  in Lemma 3.1 we clearly have

$$f(y) \asymp C_{23} G_{B(x_0, 5/8)}^V \mathbf{1}(y) \int_{B(x_0, 3/4)^c} f(z) \nu(z - x_0) dz, \quad |y - x_0| < \frac{1}{2}.$$

Thus it suffices to see that  $G_{B(x_0, 1)}^V \mathbf{1}(y) \leq C G_{B(x_0, 5/8)}^V \mathbf{1}(y)$ ,  $y \in B(x_0, 1/2)$ , with a constant  $C = C(X)$  (independent of  $V$  and  $x_0$ ). By formula (3.3) for  $D' = B(x_0, 1)$ ,  $D = B(x_0, 5/8)$  and  $f = \mathbf{1}$ , and by the fact that  $\sup_{y \in B(x_0, 1)} G_{B(x_0, 1)}^V \mathbf{1}(y) = C^{(1)} < \infty$  with  $C^{(1)}$  independent of  $V$  and  $x_0$ , we have

$$G_{B(x_0, 1)}^V \mathbf{1}(y) \leq G_{B(x_0, 5/8)}^V \mathbf{1}(y) + C^{(1)} \mathbf{E}^y[e_V(\tau_{B(x_0, 5/8)})], \quad |y - x_0| < \frac{1}{2}.$$

Let now

$$g(y) = \begin{cases} \mathbf{E}^y[e_V(\tau_{B(x_0, 5/8)})] & \text{if } y \in B(x_0, 5/8) \\ \mathbf{1} & \text{if } y \notin B(x_0, 5/8). \end{cases}$$

By applying Lemma 3.1 for  $g$  with  $r = 1/2$ ,  $p = 17/32$ ,  $q = 9/16$  and  $R = 5/8$ , we conclude that

$$\mathbf{E}^y[e_V(\tau_{B(x_0, 5/8)})] \leq C_{23} G_{B(x_0, 17/32)}^V \mathbf{1}(y) \int_{B(0, 9/16)^c} \nu(z) dz \leq C G_{B(x_0, 5/8)}^V \mathbf{1}(y), \quad |y - x_0| < \frac{1}{2},$$

with constant  $C = C(X)$ , independent of  $V$  and  $x_0$ .  $\square$

In fact, in order to obtain the above corollary it suffices to prove Lemma 3.1 only for two fixed sets of parameters  $r, p, q, R$ . Therefore, it would be enough to have Assumptions 2.1 (1) and 2.3 in place only for some specially chosen, sufficiently small  $r > 0$  and  $p, q > 0$ , respectively. However, this approach requires a detailed analysis of constants appearing in [14] and causes further technical difficulties (note that the parameters  $r, p, q$  in Lemma 3.1 do not correspond directly to  $r$  and  $p, q$  in the assumptions). Since the general Assumptions 2.1 (1) and 2.3 are not restrictive for our further results, we included a general version of Lemma 3.1.

The following auxiliary results will also be used later.

**Lemma 3.2.** *Let  $D \subset \mathbf{R}^d$  be an arbitrary open set and  $V$  be an  $X$ -Kato class potential such that  $V \geq 0$  on  $D$ . Then there are constants  $C_{25} = C_{25}(X, V, t)$  and  $C_{26} = C_{26}(X, V, t)$  such that for every  $t > 0$  we have*

- (1)  $\mathbf{E}^x \left[ \frac{t}{2} \geq \tau_D; e_V(t) \right] \leq C_{25} \mathbf{E}^x \left[ e_V(\tau_D) T_{t/2} \mathbf{1}(X_{\tau_D}) \right]$
- (2)  $\mathbf{E}^x \left[ \frac{t}{2} < \tau_D; e_V(t) \right] \leq C_{26} G_D^V \mathbf{1}(x) \sup_{y \in D} T_{t/2} \mathbf{1}(y), \quad x \in D.$

*Proof.* The proof of (1) and (2) with  $G_D^V \mathbf{1}(x)$  replaced by  $\mathbf{E}^x \left[ \frac{t}{4} < \tau_D; e_V \left( \frac{t}{4} \right) \right]$  runs in the same way as in [47, Lemma 4.3]. We complete the proof of (2) by the simple observation that

$$\mathbf{E}^x \left[ \frac{t}{4} < \tau_D; e_V \left( \frac{t}{4} \right) \right] \leq \frac{4}{t} \mathbf{E}^x \left[ \frac{t}{4} < \tau_D; \int_0^{\frac{t}{4}} e^{-\int_0^v V(X_s) ds} dv \right] \leq \frac{4}{t} G_D^V \mathbf{1}(x).$$

□

A short proof of the following fact was communicated to us by M. Kwaśnicki.

**Lemma 3.3.** *Let  $(X_t)_{t \geq 0}$  be a Lévy process with transition densities  $p(t, x, y) = p(t, y - x)$  such that for some  $t > 0$  and all  $s \in (0, t]$  we have  $p(s, x) \leq p(s, y)$  whenever  $|x| \geq |y|$ . Then for every bounded open set  $D \subset \mathbf{R}^d$  and  $r > 0$  such that  $\{y \in D : \text{dist}(y, \partial D) \geq r\} \neq \emptyset$  there is a constant  $C_{27} = C_{27}(r)$  such that*

$$p_D(t, x, y) \geq p(t, y - x) - C_{27}, \quad x, y \in D, \quad \text{dist}(y, \partial D) \geq r.$$

*Proof.* Observe that for every  $|z| \geq r$  and  $s \in (0, t]$  we have

$$|B(0, r)| p(s, z) \leq \int_{B(0, r)} p(s, w) dw \leq 1.$$

Thus by Hunt's formula (2.1) we get

$$p_D(t, x, y) = p(t, y - x) - \mathbf{E}^x [\tau_D < t; p(t - \tau_D, y - X_{\tau_D})] \geq p(t, y - x) - C_{27},$$

for all  $x, y \in D$  such that  $\text{dist}(y, \partial D) \geq r$ , with  $C_{27} = |B(0, r)|^{-1}$ . □

### 3.2. Jump estimates

For our purposes below we will need to control jumps between some carefully chosen regions. Let  $n, k \in \mathbf{N}$ ,  $n, k \geq n_0 \geq 2$  (with  $n_0$  to be chosen below), and define

$$\begin{aligned} D_n &:= \left\{ x \in \mathbf{R}^d : n - 2 < |x| \right\}, \quad n \geq n_0 + 2 \\ D_{n_0} &= D_{n_0+1} := \mathbf{R}^d \\ R_k &:= \left\{ x \in \mathbf{R}^d : k - 1 < |x| \leq k \right\}, \quad k \geq n_0 + 2 \\ R_{n_0} &:= \left\{ x \in \mathbf{R}^d : |x| \leq n_0 \right\}, \quad R_{n_0+1} := \left\{ x \in \mathbf{R}^d : |x| \leq n_0 + 1 \right\}. \end{aligned}$$

We will use the two stopping times

$$\begin{aligned} \tau_n &= \tau_{D_n} := \inf \{ t \geq 0 : X_t \notin D_n \} \\ \sigma_k &= \sigma_{R_k} := \inf \{ t \geq 0 : X_t \in R_k \}. \end{aligned}$$

Note that  $\tau_{n_0} = \tau_{n_0+1} = \infty$ . In the events in which we are interested the process jumps from the complement of a ball  $D_n$  to a smaller shell  $R_k$ , which we will refer to as admissible jumps. We define for  $k \geq n_0$ ,  $n \geq k + 2$  and  $t > 0$  the events

$$\begin{aligned} S(n, k, 1, t) &= \{ X_{\tau_n} \in R_k, \sigma_k < t \} \\ S(n, k, l, t) &= \bigcup_{p=k+2}^{n-2} S(n, p, l-1, t) \cap S(p, k, 1, t), \quad l > 1. \end{aligned}$$

The first corresponds to the event that the process arrives in shell  $R_k$  before time  $t$  in just one jump after exiting  $D_n$ . The second event is defined inductively. Let  $k + 2 \leq p \leq n - 2$ . The event

$S(n, p, 1, t) \cap S(p, k, 1, t)$  means that the process jumps to shell  $R_p$  on leaving  $D_n$  and then again jumps to shell  $R_k$  on leaving  $D_p$ , and all this occurs before time  $t$ . Note that the process may go elsewhere after arriving in  $R_p$  but the events which we are constructing only take account of admissible jumps, i.e., those that are oriented to the origin through jumps into the shells  $R_k$ . Thus the event  $S(n, k, l, t)$  corresponds to the process arriving in shell  $R_k$  from  $D_n$  through  $l$  admissible jumps before time  $t$ . This scheme of keeping track of the so defined jumps has been first devised in [17] and efficiently adopted in [57]. Here we also partially adopt the notation of [57].

The following technical lemma will be needed below.

**Lemma 3.4.** *Let Assumption 2.1 (1)-(2) be satisfied, and take  $n, k \in \mathbf{N}$  such that  $n - 2 > k \geq n_0$ . Then the following hold:*

- (1) *There is a constant  $C_{28} = C_{28}(X) \geq 1$  (independent of  $n$  and  $k$ ) such that*

$$\int_{R_k} \nu(z - y) dz \leq C_{28} \int_{R_k} \nu(z - x) dz, \quad x \in R_n, \quad y \in D_n.$$

- (2) *For any  $m \in \mathbf{N}$  there is a constant  $C_{29} = C_{29}(X, m) \geq 1$  (independent of  $k$ ) such that*

$$\int_{R_{k+m} \cap \{z: |y-z| > 1/2\}} \nu(z - y) dz \leq C_{29} \int_{R_k} \nu(z - y) dz, \quad |y| \geq k + 1.$$

*Proof.* First we prove (1). By rotation symmetry of  $R_k$  and conditions (1)-(2) of Assumption 2.1 we deduce directly that

$$\int_{R_k} \nu(z - y) dz \leq C_1^2 \int_{R_k} \nu(z - x) dz, \quad x \in R_n, \quad n - 2 < |y| \leq |x|,$$

and

$$\int_{R_k} \nu(z - y) dz \leq C_2 \int_{R_k} \nu(z - x) dz, \quad x \in R_n, \quad |x| \leq |y|,$$

respectively. Thus (1) follows. Consider assertion (2) of the lemma. Define the dilations  $S_{k,m}(w) = ((k + m - 1)/(k - 1))w$ ,  $m \in \mathbf{N}$ . Since  $R_{k+m} \subset S_{k,m}(R_k)$ , it suffices to prove (2) for  $R_{k+m}$  replaced by  $S_{k,m}(R_k)$ . By changing variables we obtain

$$\begin{aligned} \int_{S_{k,m}(R_k) \cap \{z: |y-z| > 1/2\}} \nu(z - y) dz \\ = \left( \frac{k + m - 1}{k - 1} \right)^d \int_{R_k \cap S_{k,m}^{-1}(\{z: |y-z| > 1/2\})} \nu \left( \frac{k + m - 1}{k - 1} w - y \right) dw, \end{aligned}$$

for all  $|y| \geq k + 1$ . Since  $\left| \frac{k+m-1}{k-1} w - w \right| \leq 2m$ , by Assumption 2.1 (1) we have

$$\nu \left( \frac{k + m - 1}{k - 1} w - y \right) \leq C^{(1)} \nu(w - y),$$

with  $w \in R_k \cap S_{k,m}^{-1}(\{z: |y-z| > 1/2\})$  and constant  $C^{(1)} = C^{(1)}(X, m)$ . Hence

$$\int_{S_{k,m}(R_k) \cap \{z: |y-z| > 1/2\}} \nu(z - y) dz \leq C^{(1)}(m + 1)^d \int_{R_k} \nu(w - y) dw, \quad |y| \geq k + 1,$$

which completes the proof of the lemma.  $\square$

The next two lemmas are basic tools for our further considerations. Lemma 3.5 essentially generalizes a similar result in [57, Lemma 5.7] which was obtained for rotationally symmetric relativistic stable process. This improvement was possible due to a new and more general approach which is based on a sufficiently sharp uniform upper estimate for local maxima of  $(X, V)$ -harmonic functions

(Corollary 3.1) and an inductive procedure which substantially uses Assumption 2.1 (3). Recall that a sufficiently general version of this uniform estimate necessary for our purposes in this paper was proved only recently in [14]. The second Lemma 3.6 is a relatively easy corollary of Lemma 3.5 and Assumption 2.1. Notice that it is crucial for our further applications that the constants  $C_{30}$  and  $C_{31}$  in these lemmas are independent of  $t$ , unlike in [57]. This allows us to use them in proving estimates of  $\lambda$ -subaveraging functions and, in consequence, the bounds on the eigenfunctions. Both proofs below clearly show the significance of condition (3) in Assumption 2.1.

We note for later use that under the condition (1) in Assumption 2.1 we have  $\mathbf{P}^x(\tau_n < t) > 0$  and  $\mathbf{P}^x(\sigma_k < t) > 0$  for all  $n - 2 \geq k \geq n_0$ ,  $n - 1 < |x| \leq n$  and  $t > 0$ .

**Lemma 3.5.** *Let Assumptions 2.1–2.3 hold, and  $n, k \in \mathbf{N}$  be such that  $n - 2 \geq k \geq n_0$ . Then there is a constant  $C_{30} = C_{30}(X)$  and  $\theta_0 = \theta_0(X) \geq 1$  such that for every  $t > 0$ , for all  $n - 1 < |x| \leq n$  and  $\theta > \theta_0$  we have*

$$\mathbf{E}^x \left[ \tau_n < t, X_{\tau_n} \in R_k; e^{-\theta \tau_n} \right] \leq \frac{C_{30}}{\theta} \int_{R_k} \nu(y - x) dy.$$

*Proof.* First we assume that  $n > k + 2$  and  $\theta > 0$  is arbitrary. Note that  $\text{dist}(D_n, R_k) \geq 1$ . For  $r > n - 2$  denote  $\tau_{n,r} := \tau_{D_n \cap B(0,r)}$ . Using the Ikeda-Watanabe formula [43, Theorem 1], we have

$$\begin{aligned} \mathbf{E}^x \left[ \tau_{n,r} < t, X_{\tau_{n,r}} \in R_k; e^{-\theta \tau_{n,r}} \right] \\ \leq \int_{D_n \cap B(0,r)} \int_0^\infty e^{-\theta s} p_{D_n \cap B(0,r)}(s, x, y) \int_{R_k} \nu(z - y) dz ds dy, \end{aligned}$$

and, consequently, by Lemma 3.4 (1) and Fubini's theorem we get

$$\mathbf{E}^x \left[ \tau_{n,r} < t, X_{\tau_{n,r}} \in R_k; e^{-\theta \tau_{n,r}} \right] \leq C \int_0^\infty e^{-\theta s} ds \int_{R_k} \nu(z - x) dz = \frac{C}{\theta} \int_{R_k} \nu(z - x) dz,$$

with constant  $C = C(X)$ . To complete the proof in this case it suffices to show that

$$\mathbf{E}^x \left[ \tau_{n,r} < t, X_{\tau_{n,r}} \in R_k; e^{-\theta \tau_{n,r}} \right] \rightarrow \mathbf{E}^x \left[ \tau_n < t, X_{\tau_n} \in R_k; e^{-\theta \tau_n} \right]$$

as  $r \rightarrow \infty$ . Since  $\tau_{n,r} = \tau_n$  when  $X_{\tau_{n,r}} \in R_k$ , we have

$$\begin{aligned} 0 &\leq \mathbf{E}^x \left[ \tau_n < t, X_{\tau_n} \in R_k; e^{-\theta \tau_n} \right] - \mathbf{E}^x \left[ \tau_{n,r} < t, X_{\tau_{n,r}} \in R_k; e^{-\theta \tau_{n,r}} \right] \\ &= \mathbf{E}^x \left[ \tau_{n,r} < \tau_n < t, X_{\tau_n} \in R_k, X_{\tau_{n,r}} \in B(0, r)^c; e^{-\theta \tau_n} \right] \\ &\leq \mathbf{P}^x \left( \tau_{n,r} < t, X_{\tau_{n,r}} \in B(0, r)^c \right) \\ &\leq \mathbf{P}^x \left( \tau_{B(0,r)} < t \right) \leq \mathbf{P}^0 \left( \tau_{B(0,r/2)} < t \right), \end{aligned}$$

for sufficiently large  $r$ . Clearly,  $\mathbf{P}^0(\tau_{B(0,r/2)} < t) \rightarrow 0$  as  $r \rightarrow \infty$ ,  $t > 0$ , and the claimed convergence follows.

Now consider the case  $n = k + 2$ . Denote  $B_y = B(y, 1)$  and

$$f(y) = \begin{cases} \mathbf{E}^y \left[ \tau_n < \infty, X_{\tau_n} \in R_k; e^{-\theta \tau_n} \right] & \text{if } y \in D_n \\ \mathbf{1}_{R_k}(y) & \text{if } y \notin D_n. \end{cases}$$

Since  $f$  is an  $(X, \theta)$ -regular harmonic function in  $D_n$ , we have

$$f(z) = \mathbf{E}^z \left[ e^{-\theta \tau_{B_y}} f(X_{\tau_{B_y}}) \right], \quad z \in B_y, \quad |y| > n - 1.$$



We will show that there is a constant  $C = C(X)$  and  $\theta_0 \geq 1$  such that for every  $\theta \geq \theta_0$  and  $l \in \mathbf{N}$

$$(3.5) \quad f(y) \leq \frac{C}{\theta} \left( \sum_{i=1}^l \frac{1}{2^i} \int_{R_k} \nu(z-y) dz + \frac{1}{2^l} \int_{|z|>n+1} \nu(z-y) f(z) dz \right), \quad n-1 < |y| \leq n.$$

If this holds, then by taking the limit  $l \rightarrow \infty$  the required bound follows.

By Corollary 3.1 we have

$$f(z) \leq C_{24} G_{B_y}^\theta \mathbf{1}(z) \int_{B(y, 1/2)^c} f(w) \nu(w-y) dw, \quad |z-y| < 1/2, \quad n-1 < |y|,$$

and, since  $G_{B_y}^\theta \mathbf{1}(z) \leq 1/\theta$ , we obtain

$$(3.6) \quad f(y) \leq \frac{2C_{24}}{2\theta} \left( \int_{R_k} \nu(z-y) dz + \int_{|z-y|>1/2, |z|>n-2} \nu(z-y) f(z) dz \right), \quad n-1 < |y|.$$

Moreover,  $0 \leq f \leq 1$  and a direct application of Lemma 3.4 (2) to each of the three integrals (recall that  $k = n-2$ )

$$\int_{R_{k+m} \cap \{z: |z-y|>1/2\}} \nu(z-y) dz, \quad m = 1, 2, 3,$$

separately gives that

$$(3.7) \quad f(y) \leq \frac{C}{2\theta} \left( \int_{R_k} \nu(z-y) dz + \int_{|z-y|>1/2, |z|>n+1} \nu(z-y) f(z) dz \right), \quad n-1 < |y|,$$

with constant  $C = C(X)$ . In particular, (3.5) holds for  $l = 1$  and arbitrary  $\theta > 0$ .

Next we use induction. Let  $\theta_0 := CC_3 \vee 1$  and suppose that (3.5) is true for  $l-1 \in \mathbf{N}$  with constant  $C$  and  $\theta \geq \theta_0$ , where  $C$  is the constant in (3.7). By the induction hypothesis and (3.7), we have for  $n-1 < |y| \leq n$  and  $\theta \geq \theta_0$

$$\begin{aligned} f(y) &\leq \frac{C}{\theta} \left( \sum_{i=1}^{l-1} \frac{1}{2^i} \int_{R_k} \nu(z-y) dz + \frac{1}{2^{l-1}} \int_{|z|>n+1} \nu(z-y) f(z) dz \right) \\ &\leq \frac{C}{\theta} \sum_{i=1}^{l-1} \frac{1}{2^i} \int_{R_k} \nu(z-y) dz + \left( \frac{C}{\theta} \right)^2 \frac{1}{2^l} \int_{|z|>n+1} \nu(z-y) \int_{R_k} \nu(w-z) dw dz \\ &\quad + \left( \frac{C}{\theta} \right)^2 \frac{1}{2^l} \int_{|z|>n+1} \nu(z-y) \int_{|w-z|>1/2, |w|>n+1} \nu(w-z) f(w) dw dz. \end{aligned}$$

An application of Fubini's theorem and Assumption 2.1 (3) to the last two terms in the sum on the right hand side above gives

$$f(y) \leq \frac{C}{\theta} \sum_{i=1}^l \frac{1}{2^i} \int_{R_k} \nu(w-y) dw + \frac{C}{\theta} \frac{1}{2^l} \int_{|w|>n+1} \nu(w-y) f(w) dw, \quad n-1 < |y| \leq n.$$

□

**Lemma 3.6.** *Let Assumptions 2.1–2.4 hold. Moreover, suppose that there is a non-decreasing sequence  $g_n \rightarrow \infty$  as  $n \rightarrow \infty$  and  $n_0 \in \mathbf{N}$  large enough such that for  $n \geq n_0$  we have*

$$1 < 2\theta_0 \leq g_n \leq \inf_{|y| \geq n} V(y) \quad \text{and} \quad 4C_3 C_{28} C_{30} \leq g_{n_0},$$

where  $C_{28}, C_{30}$  are constants, and  $\theta_0$  is the parameter from Lemmas 3.4 (1) and 3.5, respectively. Then for  $n-1 < |x| \leq n$ ,  $n_0 \leq k \leq n-2$ ,  $n, k, l \in \mathbf{N}$ , it follows that

$$(3.8) \quad \mathbf{E}^x \left[ S(n, k, l, t); e^{-\frac{1}{2} \int_0^{\sigma_k} V(X_s) ds} \right] \leq \frac{C_{31}}{2^l g_{n-2}} \int_{R_k} \nu(y-x) dy, \quad \text{with } C_{31} = 4C_{30}.$$

*Proof.* We use induction in  $l \in \mathbf{N}$ . Let  $l = 1$ . Since  $S(n, k, 1, t) = \{X_{\tau_n} \in R_k, \sigma_k < t\}$  and  $\tau_n = \sigma_k$  for  $X_{\tau_n} \in R_k$ , we obtain by Lemma 3.5

$$\begin{aligned} \mathbf{E}^x \left[ S(n, k, 1, t); e^{-\frac{1}{2} \int_0^{\sigma_k} V(X_s) ds} \right] &\leq \mathbf{E}^x \left[ X_{\tau_n} \in R_k, \tau_n < t; e^{-\tau_n g_{n-2}/2} \right] \\ &\leq \frac{C_{31}}{2g_{n-2}} \int_{R_k} \nu(y-x) dy. \end{aligned}$$

Let now  $l \geq 2$  and suppose that the bound (3.8) holds for  $1, 2, \dots, l-1$  and all  $n, k$  as in the statement of the lemma. The strong Markov property gives

$$\begin{aligned} \mathbf{E}^x \left[ S(n, k, l, t); e^{-\frac{1}{2} \int_0^{\sigma_k} V(X_s) ds} \right] &= \sum_{p=k+2}^{n-2} \mathbf{E}^x \left[ S(n, p, l-1, t), S(p, k, 1, t); e^{-\frac{1}{2} \int_0^{\sigma_p} V(X_s) ds} e^{-\frac{1}{2} \int_{\sigma_p}^{\sigma_k} V(X_s) ds} \right] \\ &\leq \sum_{p=k+2}^{n-2} \mathbf{E}^x \left[ S(n, p, l-1, t), S(p, k, 1, t + \sigma_p); e^{-\frac{1}{2} \int_0^{\sigma_p} V(X_s) ds} e^{-\frac{1}{2} \int_{\sigma_p}^{\sigma_k} V(X_s) ds} \right] \\ &\quad \sum_{p=k+2}^{n-2} \mathbf{E}^x \left[ S(n, p, l-1, t); e^{-\frac{1}{2} \int_0^{\sigma_p} V(X_s) ds} \mathbf{E}^{X_{\sigma_p}} \left[ S(p, k, 1, t); e^{-\frac{1}{2} \int_0^{\sigma_k} V(X_s) ds} \right] \right]. \end{aligned}$$

By the induction hypothesis and Lemma 3.4 (1) the last sum is bounded above by

$$\sum_{p=k+2}^{n-2} \frac{C_{31}}{2^{l-1} g_{n-2}} \int_{R_p} \nu(y-x) \frac{C_{31} C_{28}}{2g_{p-2}} \int_{R_k} \nu(z-y) dz dy.$$

Hence, Fubini's theorem and Assumption 2.1 (3) give that

$$\begin{aligned} \mathbf{E}^x \left[ S(n, k, l, t); e^{-\frac{1}{2} \int_0^{\sigma_k} V(X_s) ds} \right] &\leq \frac{C_{31}}{2^l g_{n-2}} \frac{C_{31} C_{28}}{g_{n_0}} \int_{R_k} \sum_{p=k+2}^{n-2} \int_{R_p} \nu(y-x) \nu(z-y) dy dz \\ &\leq \frac{C_{31}}{2^l g_{n-2}} \frac{4C_{30} C_{28} C_3}{g_{n_0}} \int_{R_k} \nu(y-x) dy \\ &\leq \frac{C_{31}}{2^l g_{n-2}} \int_{R_k} \nu(y-x) dy. \end{aligned}$$

□

### 3.3. Estimates of $\lambda$ -subaveraging functions

*Proof of Theorem 2.1.* Recall that  $C_{21} = C_{21}(X, V)$ ,  $C_{22} = C_{22}(X, V)$  and  $C_{29} = C_{29}(X, m)$  are the constants in (3.1) and Lemma 3.4 (2), respectively. For the rest of the proof we set:

$$\begin{aligned} C &= C(X) := 2 \left( 1 \vee \frac{C_2}{|B(0,1)| \nu((6,0,\dots,0))} \right) \geq 2 \\ C^{(1)} &= C^{(1)}(X) := \max_{1 \leq m \leq 2} C_{29} \geq 1 \\ C^{(2)} &= C^{(2)}(X) := \frac{1}{4} \left( 1 \wedge \left( \int_{B(0,1)^c} \nu(y) dy \right)^{-1} \right) \leq \frac{1}{4}. \end{aligned}$$

It is worth to note that  $C^{(2)} \int_{B(0,1)^c} \nu(y) dy \leq 1/4$ . For  $n \in \mathbf{N}$  also denote  $g_n := \inf_{|y| \geq n} V_0(y)$  with  $V_0 = V - \lambda$ . Let  $n_0$  be a natural number satisfying the assumptions of Lemma 3.6 for the potential  $V_0$  and the sequence  $(g_n)_{n \in \mathbf{N}}$ , and such that

$$(3.9) \quad \max \left( C_{31} \left( \left( \int_{B(0,1)^c} \nu(y) dy \vee C_2 C_3 \right) + \frac{6C_{21} C C_{28} C^{(1)}}{C^{(2)}} \right), 2(\lambda + C_{22}) \right) \leq g_{n_0},$$

where  $C_{31}$  is the constant from Lemma 3.6. We will show that

$$(3.10) \quad \varphi(x) \leq C \|\varphi\|_\infty \left( \int_{R_{n_0}} \nu(y-x) dy \right)^{\sum_{i=1}^p 2^{-i}} \quad \text{for } |x| \geq n_0 + 3,$$

for all  $p \in \mathbf{N}$ . If this holds, then by taking the limit  $p \rightarrow \infty$  and using Assumption 2.1 (1) the theorem follows.

We use again induction. For more clarity we divide the proof of (3.10) in two steps.

*Step 1:* First we show that the bound (3.10) holds for  $p = 1$ . We have

$$\varphi(x) \leq e^{\lambda t} \mathbf{E}^x \left[ e^{-\int_0^t V(X_s) ds} \varphi(X_t) \right] \leq \|\varphi\|_\infty \mathbf{E}^x \left[ e^{-\int_0^t V_0(X_s) ds} \right], \quad x \in \mathbf{R}^d, \quad t > 0.$$

To get an upper bound on the latter expectation, let  $n-1 \leq |x| < n$ ,  $n \geq n_0 + 4$ . For all  $t > 0$  we have

$$(3.11) \quad \begin{aligned} \mathbf{E}^x \left[ e^{-\int_0^t V_0(X_s) ds} \right] &\leq \mathbf{E}^x \left[ \tau_n > t; e^{-\int_0^t V_0(X_s) ds} \right] \\ &+ \sum_{k=n_0}^{n-2} \sum_{l=1}^{\infty} \mathbf{E}^x \left[ S(n, k, l, t), \tau_k > t; e^{-\int_0^t V_0(X_s) ds} \right]. \end{aligned}$$

Clearly, by (3.9) the first term at the right hand side is estimated directly by

$$\mathbf{P}^x(\tau_n > t) e^{-2(\lambda + C_{22})t} \leq e^{-(\lambda + C_{22})t}.$$

Lemma 3.6 and (3.9) yield for  $k \geq n_0 + 2$

$$\begin{aligned} \mathbf{E}^x \left[ S(n, k, l, t), \tau_k > t; e^{-\int_0^t V_0(X_s) ds} \right] &\leq \mathbf{E}^x \left[ S(n, k, l, t), \tau_k > t; e^{-\frac{1}{2} \int_0^{\sigma_k} V_0(X_s) ds} e^{-\frac{1}{2} \int_0^t V_0(X_s) ds} \right] \\ &\leq e^{-(\lambda + C_{22})t} \mathbf{E}^x \left[ S(n, k, l, t); e^{-\frac{1}{2} \int_0^{\sigma_k} V_0(X_s) ds} \right] \\ &\leq \frac{C_{31}}{g_{n_0} 2^l} e^{-(\lambda + C_{22})t} \int_{R_k} \nu(y-x) dy. \end{aligned}$$

Similarly, by the strong Markov property, (3.1), Lemma 3.6 and Lemma 3.4 (2), we have for  $k = n_0$  and  $k = n_0 + 1$  (recall  $\tau_{n_0} = \tau_{n_0+1} = \infty$ )

$$\begin{aligned}
& \mathbf{E}^x \left[ S(n, k, l, t), \tau_k > t; e^{-\int_0^t V_0(X_s) ds} \right] \\
& \leq e^{\lambda t} \mathbf{E}^x \left[ S(n, k, l, t); e^{-\int_0^{\sigma_k} V_+(X_s) ds} e^{\int_0^{t+\sigma_k} V_-(X_s) ds} \right] \\
& = e^{\lambda t} \mathbf{E}^x \left[ S(n, k, l, t); e^{-\int_0^{\sigma_k} V(X_s) ds} e^{\int_{\sigma_k}^{t+\sigma_k} V_-(X_s) ds} \right] \\
& = e^{\lambda t} \mathbf{E}^x \left[ S(n, k, l, t); e^{-\int_0^{\sigma_k} V(X_s) ds} \mathbf{E}^{X_{\sigma_k}} \left[ e^{\int_0^t V_-(X_s) ds} \right] \right] \\
& \leq C_{21} e^{(\lambda + C_{22})t} \mathbf{E}^x \left[ S(n, k, l, t); e^{-\frac{1}{2} \int_0^{\sigma_k} 2V(X_s) ds} \right] \\
& \leq \frac{C_{21} C_{31}}{g_{n_0} 2^l} e^{(\lambda + C_{22})t} \int_{R_k} \nu(y - x) dy \\
& \leq \frac{2C_{21} C_{31} C^{(1)}}{g_{n_0} 2^l} e^{(\lambda + C_{22})t} \int_{R_{n_0}} \nu(y - x) dy.
\end{aligned}$$

By putting together the above estimates and choosing

$$t = -\frac{1}{2(\lambda + C_{22})} \log \left( C^{(2)} \int_{R_{n_0}} \nu(y - x) dy \right) > 0$$

in (3.11), and using (3.9) we conclude that

$$\begin{aligned}
\varphi(x) & \leq \|\varphi\|_\infty \left( \left( C^{(2)} \int_{R_{n_0}} \nu(y - x) dy \right)^{1/2} \right. \\
& \quad + \frac{C_{31}}{g_{n_0}} \left( C^{(2)} \int_{R_{n_0}} \nu(y - x) dy \right)^{1/2} \sum_{k=n_0+2}^{n-2} \int_{R_k} \nu(y - x) dy \\
& \quad \left. + \frac{4C_{21} C_{31} C^{(1)}}{\sqrt{C^{(2)}} g_{n_0}} \left( \int_{R_{n_0}} \nu(y - x) dy \right)^{1/2} \right) \\
& \leq \|\varphi\|_\infty \left( 1 + \frac{C_{31}}{g_{n_0}} \left( \int_{B(0,1)^c} \nu(y) dy + \frac{4C_{21} C^{(1)}}{\sqrt{C^{(2)}}} \right) \right) \left( \int_{R_{n_0}} \nu(y - x) dy \right)^{1/2} \\
& \leq 2 \|\varphi\|_\infty \left( \int_{R_{n_0}} \nu(y - x) dy \right)^{1/2},
\end{aligned}$$

which completes the first step.

*Step 2:* Suppose now that (3.10) holds for  $p$ . We prove that it is also satisfied for  $p+1$ . We consider two cases. First let  $n_0 + 3 \leq |x| < n_0 + 4$ . Denote  $x_0 = ((n_0 - 1)/|x|)x$ . Then by the fact that  $1 \leq |y - x| \leq 6$  for  $y \in B(x_0, 1)$  and by Assumption 2.1 (2), we have

$$\frac{\nu((6, 0, \dots, 0)|B(0, 1))}{C_2} \leq \int_{B(x_0, 1)} \nu(y - x) dy \leq \int_{R_{n_0}} \nu(y - x) dy.$$

By using this estimate and the definition of  $C$ , it is immediate to obtain the bound

$$\varphi(x) \leq \|\varphi\|_\infty \leq C \|\varphi\|_\infty \left( \int_{R_{n_0}} \nu(y - x) dy \right)^{\sum_{i=1}^{p+1} 2^{-i}}.$$

Let now  $n - 1 \leq |x| < n$ ,  $n \geq n_0 + 5$ . Similarly as before, for all  $t > 0$  we have

$$(3.12) \quad \begin{aligned} \varphi(x) &\leq \mathbf{E}^x \left[ \tau_n > t; e^{-\int_0^t V_0(X_s) ds} \varphi(X_t) \right] \\ &+ \sum_{k=n_0}^{n-2} \sum_{l=1}^{\infty} \mathbf{E}^x \left[ S(n, k, l, t), \tau_k > t; e^{-\int_0^t V_0(X_s) ds} \varphi(X_t) \right]. \end{aligned}$$

By (3.10), (3.9) and Lemma 3.4 (1), the first term on the right hand side of (3.12) is bounded above by

$$\begin{aligned} e^{-2(\lambda+C_{22})t} \sup_{|z|>n-2} \varphi(z) &\leq e^{-(\lambda+C_{22})t} C \|\varphi\|_{\infty} \sup_{|z|>n-2} \left( \int_{R_{n_0}} \nu(y-z) dy \right)^{\sum_{i=1}^p 2^{-i}} \\ &\leq e^{-(\lambda+C_{22})t} C C_{28} \|\varphi\|_{\infty} \left( \int_{R_{n_0}} \nu(y-x) dy \right)^{\sum_{i=1}^p 2^{-i}}. \end{aligned}$$

By the same arguments, Lemmas 3.4 (1) and 3.6, Assumption 2.1 (2) and Fubini's theorem we also have for  $k \geq n_0 + 3$

$$\begin{aligned} &\mathbf{E}^x \left[ S(n, k, l, t), \tau_k > t; e^{-\int_0^t V_0(X_s) ds} \varphi(X_t) \right] \\ &\leq \sup_{|z|>k-2} \varphi(z) \mathbf{E}^x \left[ S(n, k, l, t), \tau_k > t; e^{-\frac{1}{2} \int_0^{\sigma_k} V_0(X_s) ds} e^{-\frac{1}{2} \int_0^t V_0(X_s) ds} \right] \\ &\leq C e^{-(\lambda+C_{22})t} \|\varphi\|_{\infty} \sup_{|z|>k-2} \left( \int_{R_{n_0}} \nu(y-z) dy \right)^{\sum_{i=1}^p 2^{-i}} \mathbf{E}^x \left[ S(n, k, l, t); e^{-\frac{1}{2} \int_0^{\sigma_k} V_0(X_s) ds} \right] \\ &\leq \frac{C C_{31}}{g_{n_0} 2^l} e^{-(\lambda+C_{22})t} \|\varphi\|_{\infty} \sup_{|z|>k-2} \left( \int_{R_{n_0}} \nu(y-z) dy \right)^{1-2^{-p}} \int_{R_k} \nu(z-x) dz \\ &\leq \frac{C C_{31} C_2 C_{28}}{g_{n_0} 2^l} \|\varphi\|_{\infty} e^{-(\lambda+C_{22})t} \left( \int_{R_{n_0}} \nu(y-x) dy \right)^{-2^{-p}} \int_{R_{n_0}} \int_{R_k} \nu(y-z) \nu(z-x) dz dy. \end{aligned}$$

Similarly, by the strong Markov property, Lemmas 3.6 and 3.4 (2), we estimate the expectations for  $n_0 \leq k \leq n_0 + 2$  to obtain

$$\begin{aligned} &\mathbf{E}^x \left[ S(n, k, l, t), \tau_k > t; e^{-\int_0^t V_0(X_s) ds} \varphi(X_t) \right] \\ &\leq \|\varphi\|_{\infty} e^{\lambda t} \mathbf{E}^x \left[ S(n, k, l, t); e^{-\int_0^{\sigma_k} V_+(X_s) ds} e^{\int_0^{t+\sigma_k} V_-(X_s) ds} \right] \\ &= \|\varphi\|_{\infty} e^{\lambda t} \mathbf{E}^x \left[ S(n, k, l, t); e^{-\int_0^{\sigma_k} V(X_s) ds} e^{\int_{\sigma_k}^{t+\sigma_k} V_-(X_s) ds} \right] \\ &= \|\varphi\|_{\infty} e^{\lambda t} \mathbf{E}^x \left[ S(n, k, l, t); e^{-\int_0^{\sigma_k} V(X_s) ds} \mathbf{E}^{X_{\sigma_k}} \left[ e^{\int_0^t V_-(X_s) ds} \right] \right] \\ &\leq C_{21} \|\varphi\|_{\infty} e^{(\lambda+C_{22})t} \mathbf{E}^x \left[ S(n, k, l, t); e^{-\frac{1}{2} \int_0^{\sigma_k} 2V(X_s) ds} \right] \\ &\leq \frac{C_{21} C_{31}}{g_{n_0} 2^l} \|\varphi\|_{\infty} e^{(\lambda+C_{22})t} \int_{R_k} \nu(y-x) dy \\ &\leq \frac{2C_{21} C_{31} C^{(1)}}{g_{n_0} 2^l} \|\varphi\|_{\infty} e^{(\lambda+C_{22})t} \int_{R_{n_0}} \nu(y-x) dy. \end{aligned}$$

Combining the above estimates and choosing

$$t = -\frac{1}{\lambda + C_{22}} \left( \log(C_{28}C)^{-1} + 2^{-(p+1)} \log \left( C^{(2)} \int_{R_{n_0}} \nu(y-x) dy \right) \right) > 0$$

in (3.12), by Assumption 2.1 (3) and (3.9) we obtain

$$\begin{aligned} \varphi(x) &\leq \|\varphi\|_\infty \left( \left( \int_{R_{n_0}} \nu(y-x) dy \right)^{\sum_{i=1}^{p+1} 2^{-i}} \right. \\ &\quad + \frac{C_{31}C_2}{g_{n_0}} \left( \int_{R_{n_0}} \nu(z-x) dz \right)^{2^{-(p+1)}-2^{-p}} \int_{R_{n_0}} \left( \sum_{k=n_0+3}^{n-2} \int_{R_k} \nu(y-z) \nu(y-x) dy \right) dz \\ &\quad \left. + \frac{6C_{21}C_{31}CC_{28}C^{(1)}}{C^{(2)}g_{n_0}} \left( \int_{R_{n_0}} \nu(y-x) dy \right)^{\sum_{i=1}^{p+1} 2^{-i}} \right) \\ &\leq \|\varphi\|_\infty \left( 1 + \frac{C_{31}C_2C_3}{g_{n_0}} + \frac{6C_{21}C_{31}CC_{28}C^{(1)}}{C^{(2)}g_{n_0}} \right) \left( \int_{R_{n_0}} \nu(y-x) dy \right)^{\sum_{i=1}^{p+1} 2^{-i}} \\ &\leq 2 \|\varphi\|_\infty \left( \int_{R_{n_0}} \nu(y-x) dy \right)^{\sum_{i=1}^{p+1} 2^{-i}} \end{aligned}$$

which completes the proof of the theorem.  $\square$

*Proof of Theorem 2.2.* By integrating in the equality  $e^{-\lambda t} \varphi(x) = \mathbf{E}^x \left[ e^{-\int_0^t V(X_s) ds} \varphi(X_t) \right]$  over  $t \in (0, \infty)$ , it follows that

$$\varphi(x) = \lambda G^V \varphi(x), \quad x \in \mathbf{R}^d,$$

and by (3.3) applied to  $f = \varphi$ ,  $D' = \mathbf{R}^d$ ,  $D = B(x, 1)$ , we obtain

$$(3.13) \quad \varphi(x) = \lambda G_D^V \varphi(x) + \mathbf{E}^x \left[ e^{-\int_0^{\tau_D} V(X_s) ds} \varphi(X_{\tau_D}) \right], \quad x \in \mathbf{R}^d.$$

We now prove assertion (1). Let  $R > 2$  be large enough so that  $V(x) \geq 0$  for  $|x| \geq R-1$  and the assertion of Theorem 2.1 for the  $\lambda$ -subaveraging function  $|\varphi|$  holds. Let  $|x| \geq R+2$ . By (3.13), we have

$$|\varphi(x)| \leq \lambda G_D^V |\varphi(x)| + \mathbf{E}^x [e_V(\tau_D) |\varphi(X_{\tau_D})|] = \text{I} + \text{II}.$$

By Theorem 2.1 applied to  $|\varphi|$ , we have

$$\text{I} \leq C G_D^V \mathbf{1}(x) \sup_{y \in D} |\varphi(y)| \leq C \|\varphi\|_\infty G_D^V \mathbf{1}(x) \nu(x),$$

with  $C = C(X, V, \lambda)$ . To estimate II first note that by Theorem 2.1, Assumption 2.1 (1), (3), the Ikeda-Watanabe formula and the fact that  $\sup_{z \in D} \mathbf{E}^z[\tau_D] \leq \mathbf{E}^0[\tau_{B(0,2)}] < \infty$ , we have for

$$z \in D \setminus B(x, 3/4)$$

(3.14)

$$\begin{aligned} \mathbf{E}^z [e_V(\tau_D) |\varphi(X_{\tau_D})|] &\leq \mathbf{E}^z [|\varphi(X_{\tau_D})|; X_{\tau_D} \in B(x, 2) \setminus D] + \mathbf{E}^z [|\varphi(X_{\tau_D})|; X_{\tau_D} \in B(x, 2)^c] \\ &\leq C \left( \|\varphi\|_\infty \nu(x) + \int_D G_D(z, y) \int_{B(x, 2)^c} |\varphi(w)| \nu(w - y) dw dy \right) \\ &\leq C \left( \|\varphi\|_\infty \nu(x) + \mathbf{E}^z[\tau_D] \int_{B(x, 2)^c} |\varphi(w)| \nu(w - x) dz \right) \\ &\leq C \|\varphi\|_\infty \left( \nu(x) + \int_{B(0, R)^c \cap B(x, 2)^c} \nu(w) \nu(w - x) dz + \int_{B(0, R)} \nu(w - x) dw \right) \\ &\leq C \|\varphi\|_\infty \nu(x), \end{aligned}$$

with  $C = C(X, V, \lambda)$ . Thus by using Corollary 3.1, the above estimate, Theorem 2.1 and Assumption 2.1 (1), (3), we finally have

$$\begin{aligned} \Pi &\leq C G_D^V \mathbf{1}(x) \left( \int_{D \cap B(x, 3/4)^c} \mathbf{E}^z [e_V(\tau_D) |\varphi(X_{\tau_D})|] \nu(z - x) dz + \int_{D^c} |\varphi(z)| \nu(z - x) dz \right) \\ &\leq C \|\varphi\|_\infty G_D^V \mathbf{1}(x) \left( \nu(x) \int_{D \cap B(x, 3/4)^c} \nu(z - x) dz + \int_{D^c \cap B(0, R)^c} \nu(z) \nu(z - x) dz \right. \\ &\quad \left. + \int_{B(0, R)} \nu(z - x) dz \right) \\ &\leq C \|\varphi\|_\infty G_D^V \mathbf{1}(x) \nu(x), \end{aligned}$$

where  $C = C(X, V, \lambda)$ . We conclude that  $|\varphi(x)| \leq C_5 \|\varphi\|_\infty G_D^V \mathbf{1}(x) \nu(x)$  for all  $|x| \geq R + 2$ , with constant  $C_5 = C_5(X, V, \lambda)$ . Now consider (2). Again, by (3.13), strict positivity of  $\varphi$  and Corollary 3.1 we have

$$\varphi(x) \geq \mathbf{E}^x \left[ e^{-\int_0^{\tau_D} V(X_s) ds} \varphi(X_{\tau_D}) \right] \geq C_{24}^{-1} G_{B(x, 1)}^V \mathbf{1}(x) \int_{B(0, 1)} \varphi(z) \nu(x - z) dz, \quad |x| \geq R.$$

By Assumption 2.1 (1), the last integral is greater than  $C \nu(x) \int_{B(0, 1)} \varphi(z) dz$  and the required inequality follows again from the positivity of  $\varphi$  with constant  $C_6 = C_6(X, \varphi)$ .  $\square$

### 3.4. Eigenfunction estimates

*Proof of Theorem 2.3.* Let  $\eta \geq 0$  be such that  $\lambda_0 + \eta > 0$  and let  $n \geq 0$  be fixed. We thus clearly have  $\varphi_n(x) = e^{\lambda t} \mathbf{E}^x [e_{V+\eta}(t) \varphi_n(X_t)]$ ,  $x \in \mathbf{R}^d$ , with  $\lambda = \lambda_n + \eta > \lambda_0 + \eta > 0$ , and the result immediately follows from Theorem 2.2 (1) for  $\varphi = \varphi_n$ .  $\square$

*Proof of Theorem 2.4.* Let  $\eta \geq 0$  be such that  $\lambda_0 + \eta > 0$ . The result directly follows from Theorems 2.3 and 2.2 (2) for  $\varphi = \varphi_0 > 0$  and  $\lambda = \lambda_0 + \eta > 0$ .  $\square$

### 3.5. Intrinsic ultracontractivity

*Proof of Theorem 2.5.* First we prove (1). By a standard argument based on the duality and symmetry of  $\tilde{T}_t$  we have  $\|\tilde{T}_t\|_{1 \rightarrow 2} = \|\tilde{T}_t\|_{2 \rightarrow \infty}$ . Since  $\|\tilde{T}_{t_0}\|_{2 \rightarrow \infty} < \infty$ , by the semigroup property it is

straightforward that also  $\|\tilde{T}_{2t_0}\|_{1 \rightarrow \infty} \leq \|\tilde{T}_{t_0}\|_{2 \rightarrow \infty}^2$ . Hence

$$\frac{e^{2\lambda_0 t_0}}{\varphi_0(x)} T_{2t_0} \mathbf{1}(x) = \tilde{T}_{2t_0} \left( \frac{1}{\varphi_0} \right) (x) \leq C_{t_0},$$

since by Theorem 2.3 we have  $1/\varphi_0 \in L^1(\mathbf{R}^d, \varphi_0^2 dx)$ . Hence the implications  $t_0$ -IUC  $\Rightarrow$   $2t_0$ -GSD and IUC  $\Rightarrow$  GSD follow. To show  $t_0$ -GSD  $\Rightarrow$   $2t_0$ -IUC ( $t_0 \geq t_b$ ), it suffices to observe that for all  $f \in L^2(\mathbf{R}^d, \varphi_0^2 dx)$  we have

$$T_{2t_0}(f\varphi_0)(x) = T_{t_0}T_{t_0}(f\varphi_0)(x) \leq T_{t_0}\mathbf{1}(x)\|T_{t_0}\|_{2 \rightarrow \infty}\|f\|_2 \leq C_{t_0}\|f\|_2\varphi_0(x).$$

The last inequality follows from  $\|T_{t_0}\|_{2 \rightarrow \infty} < \infty$ , coming from the boundedness of  $p(t_0, x)$  in  $x \in \mathbf{R}^d$ . Moreover, since GSD means  $t$ -GSD for all  $t > 0$ , assertion (2) of the theorem follows again by the latter estimate.  $\square$

**Lemma 3.7.** *Let  $V$  be a  $X$ -Kato class potential, non-negative outside a bounded subset of  $\mathbf{R}^d$ . Let Assumptions 2.1–2.3 be satisfied and consider the following two conditions:*

(1) *There exist  $C_{32} = C_{32}(X, V, t)$  and  $R > 0$  such that*

$$(3.15) \quad T_t \mathbf{1}(x) \leq C_{32} \nu(x), \quad |x| \geq R.$$

(2) *There exist  $C_{33} = C_{33}(X, V, t)$  and  $R > 0$  such that*

$$(3.16) \quad T_t \mathbf{1}(x) \leq C_{33} G_{B(x,1)}^V \mathbf{1}(x) \nu(x), \quad |x| \geq R.$$

*Statements (1) and (2) are equivalent in the following sense. If (2) is true for some  $t = s > 0$ , then (1) also follows for  $t = s$ . If (1) holds true for some  $t = s > 0$ , then (2) follows for  $t = 2s$ .*

*Proof.* For the proof of the implication (2)  $\Rightarrow$  (1) it suffices to note that there is  $R > 0$  large enough such that  $G_{B(x,1)}^V \mathbf{1}(x) \leq G_{B(x,1)}^0 \mathbf{1}(x) = \mathbf{E}^x[\tau_{B(x,1)}] = \mathbf{E}^0[\tau_{B(0,1)}] < \infty$  for  $|x| \geq R$ .

For the converse implication we suppose that (1) holds for some  $t/2 > 0$  and find  $R_1 \geq R \vee 2$  such that  $V(x) \geq 0$  for  $|x| \geq R_1 - 1$ . Denote  $D = B(x, 1)$  and let  $|x| \geq R_1 + 2$ . By Lemma 3.2 and Corollary 3.1, we have

$$\begin{aligned} T_t \mathbf{1}(x) &= \mathbf{E}^x \left[ \frac{t}{2} < \tau_D; e_V(t) \right] + \mathbf{E}^x \left[ \frac{t}{2} \geq \tau_D; e_V(t) \right] \\ &\leq C \left( G_D^V \mathbf{1}(x) \sup_{y \in D} T_{t/2} \mathbf{1}(y) \right) + \mathbf{E}^x [e_V(\tau_D) T_{t/2} \mathbf{1}(X_{\tau_D})] \\ &\leq C G_D^V \mathbf{1}(x) \left( \sup_{y \in B(x,1)} T_{t/2} \mathbf{1}(y) + \int_{B(x,1) \cap B(x,3/4)^c} \mathbf{E}^z [e_V(\tau_D) T_{t/2} \mathbf{1}(X_{\tau_D})] \nu(z-x) dz \right. \\ &\quad \left. + \int_{B(x,1)^c \cap B(0,R_1)^c} T_{t/2} \mathbf{1}(z) \nu(z-x) dz + \sup_{y \in B(0,R_1)} T_{t/2} \mathbf{1}(y) \int_{B(0,R_1)} \nu(z-x) dz \right). \end{aligned}$$

Notice that by using (1) and exactly the same arguments as in (3.14) applied to  $|\varphi(\cdot)|$  replaced by  $T_{t/2} \mathbf{1}(\cdot)$  we get

$$(3.17) \quad \mathbf{E}^z [e_V(\tau_D) T_{t/2} \mathbf{1}(X_{\tau_D})] \leq C \nu(x), \quad z \in D \cap B(x, 3/4)^c,$$

with constant  $C = C(X, V, t)$ . Thus, by estimate (1), Assumption 2.1 (1), (3) and (3.17), we conclude similarly as in the proof of Theorem 2.2 that

$$T_t \mathbf{1}(x) \leq C G_{B(x,1)}^V \mathbf{1}(x) \nu(x), \quad |x| \geq R_1 + 2,$$

with  $C = C(X, V, t)$ , which completes the proof.  $\square$



**Theorem 3.1.** *Let Assumptions 2.1–2.4 hold. If there exist a constant  $C_{13}$  and  $R > 0$  such that*

$$(3.18) \quad \frac{V(x)}{|\log \nu(x)|} \geq C_{13}, \quad |x| \geq R,$$

*then the bound (3.15) holds for all  $t \geq t_0 = 2/C_{13}$ . If, moreover,*

$$(3.19) \quad \lim_{|x| \rightarrow \infty} \frac{V(x)}{|\log \nu(x)|} = \infty,$$

*then this bound holds for every  $t > 0$ .*

*Proof.* First assume that the inequality (3.18) is satisfied for  $R > 0$ , and denote  $t_0 = 2/C_{13}$ . Choose  $n_0 \geq R$  large enough such that

$$C_2 \nu((n, 0, \dots, 0)) < 1 \quad \text{and} \quad 2\theta_0 \leq g_n := \inf_{|y| \geq n} V(y) \quad \text{for } n \geq n_0,$$

and

$$C_3 C_{28} C_{31} \leq g_{n_0},$$

where  $C_{31}$  is the constant and  $\theta_0$  is the parameter from Lemma 3.6. Thus the assumptions of Lemma 3.6 are satisfied. Moreover, by (3.18) and Assumption 2.1 (2),

$$(3.20) \quad g_n \geq -C_{13} \log(C_2 \nu((n, 0, \dots, 0))) \quad \text{for } n \geq n_0.$$

We show that for every  $t \geq t_0$  condition (3.15) holds. Let  $n-1 \leq |x| < n$ ,  $n \geq n_0 + 4$  and  $t \geq t_0$ . We have

$$(3.21) \quad T_t \mathbf{1}(x) \leq \mathbf{E}^x \left[ \tau_n > t; e^{-\int_0^t V(X_s) ds} \right] + \sum_{k=n_0}^{n-2} \sum_{l=1}^{\infty} \mathbf{E}^x \left[ S(n, k, l, t), \tau_k > t; e^{-\int_0^t V(X_s) ds} \right].$$

By (3.20) and Assumption 2.1 (1), the first term at the right hand side can be easily estimated by

$$\mathbf{P}^x(\tau_n > t) e^{-tg_{n-2}} \leq e^{C_{13}t \log(C_2 \nu((n-2, 0, \dots, 0)))} \leq C_1^2 C_2 \nu(x).$$

Similar arguments and Lemma 3.6 also yield

$$\begin{aligned} \mathbf{E}^x \left[ S(n, k, l, t), \tau_k > t; e^{-\int_0^t V(X_s) ds} \right] &\leq \mathbf{E}^x \left[ S(n, k, l, t), \tau_k > t; e^{-\frac{1}{2} \int_0^{\sigma_k} V(X_s) ds} e^{-\frac{1}{2} \int_0^t V(X_s) ds} \right] \\ &\leq e^{\frac{1}{2} C_{13} t \log(C_2 \nu((k-2, 0, \dots, 0)))} \mathbf{E}^x \left[ S(n, k, l, t); e^{-\frac{1}{2} \int_0^{\sigma_k} V(X_s) ds} \right] \\ &\leq \frac{C_1^2 C_2 C_{31}}{2^l g_{n_0}} \int_{R_k} \nu(y) \nu(y-x) dy \end{aligned}$$

for  $k \geq n_0 + 2$ . For  $k \in \{n_0, n_0 + 1\}$  we have

$$\begin{aligned} \mathbf{E}^x \left[ S(n, k, l, t), \tau_k > t; e^{-\int_0^t V(X_s) ds} \right] &\leq \mathbf{E}^x \left[ S(n, k, l, t); e^{-\int_0^{\sigma_k} V_+(X_s) ds} e^{\int_0^{t+\sigma_k} V(X_s) ds} \right] \\ &= \mathbf{E}^x \left[ S(n, k, l, t); e^{-\int_0^{\sigma_k} V(X_s) ds} e^{\int_{\sigma_k}^{t+\sigma_k} V_-(X_s) ds} \right] \\ &= \mathbf{E}^x \left[ S(n, k, l, t); e^{-\int_0^{\sigma_k} V(X_s) ds} \mathbf{E}^{X_{\sigma_k}} \left[ e^{\int_0^t V_-(X_s) ds} \right] \right] \\ &\leq C_{21} e^{C_{22}t} \mathbf{E}^x \left[ S(n, k, l, t); e^{-\frac{1}{2} \int_0^{\sigma_k} 2V(X_s) ds} \right] \\ &\leq \frac{C_{21} C_{31} e^{C_{22}t}}{2^l g_{n_0}} \int_{R_k} \nu(y-x) dy, \end{aligned}$$

by the strong Markov property, (3.1) and Lemma 3.6. Thus by Assumption 2.1 (1) and (3), the second term at the right hand side of (3.21) is bounded above by

$$\begin{aligned} & \sum_{k=n_0}^{n-2} \sum_{l=1}^{\infty} \mathbf{E}^x \left[ S(n, k, l, t), \tau_k > t; e^{-\int_0^t V(X_s) ds} \right] \\ & \leq C \sum_{l=1}^{\infty} 2^{-l} \left( \sum_{k=n_0}^{n_0+1} \int_{R_k} \nu(y-x) dy + \sum_{k=n_0+2}^{n-2} \int_{R_k} \nu(y) \nu(y-x) dy \right) \\ & \leq C \nu(x), \end{aligned}$$

where  $C = C(X, V, t)$ , and the first part of the theorem is proved. The second assertion follows from the first part by observing that (3.19) implies (3.18) with arbitrarily large constant  $C_{13}$ .  $\square$

*Proof of Theorem 2.6.* We first prove (1). By Theorem 3.1, Lemma 3.7 and Theorem 2.4 there is  $R > 0$  such that for all  $|x| > R$  condition (2.5) holds with  $t_0 = 4/C_{13}$ . The same is true for  $|x| \leq R$  by boundedness of  $T_t \mathbf{1}$ , and by continuity and strict positivity of  $\varphi_0$ .

To prove (2) fix  $\varepsilon \in (0, 1]$  and first note that for every  $t > 0$  we have  $\mathbf{P}^0(t < \tau_{B(0, \varepsilon)}) > 0$ . This positivity property follows from [68] for small  $t > 0$  and extends to all  $t > 0$ . By definition of  $t_0$ -GSD and Theorem 2.4, there is  $R > 0$  such that

$$e^{-t_0 \sup_{|y-x| < \varepsilon} V(y)} \mathbf{P}^x(t_0 < \tau_{B(x, \varepsilon)}) \leq T_{t_0} \mathbf{1}(x) \leq C \varphi_0(x) \leq C \nu(x),$$

with  $C = C(X, V, t_0)$ , for all  $|x| > R$ . Thus, by the fact that  $|\log \nu(x)| \rightarrow \infty$  as  $|x| \rightarrow \infty$ , we can choose  $R_\varepsilon \geq R$  large enough such that

$$\frac{\sup_{|y-x| < \varepsilon} V(y)}{|\log \nu(x)|} \geq \frac{1}{t_0} \left( 1 - \frac{\log \frac{C}{\mathbf{P}^0(t_0 < \tau_{B(0, \varepsilon)})}}{|\log \nu(x)|} \right) \geq \frac{1}{2t_0},$$

for  $|x| > R_\varepsilon$ , which gives the required bound.  $\square$

*Proof of Theorem 2.7.* To prove (1) observe that for any  $t > 0$  we can find  $R > 0$  such that  $V(x) \geq (4/t)|\log \nu(x)|$  for  $|x| > R$ , and we can proceed in the same way as in the proof of (1) of Theorem 2.6.

When the semigroup  $\{T_t : t \geq 0\}$  is IUC, then by the same arguments as in the proof of Theorem 2.6(2) for every  $t > 0$  there is  $C = C(X, V, t)$  such that we have

$$\frac{\sup_{|y-x| < \varepsilon} V(y)}{|\log \nu(x)|} \geq \frac{1}{t} \left( 1 - \frac{\log \frac{C}{\mathbf{P}^0(t < \tau_{B(0, \varepsilon)})}}{|\log \nu(x)|} \right), \quad |x| > R, \quad t > 0,$$

for some  $R > 0$ . Thus  $\liminf_{|x| \rightarrow \infty} \frac{\sup_{|y-x| < \varepsilon} V(y)}{|\log \nu(x)|} = \frac{1}{t}$ ,  $t > 0$ , which completes the proof.  $\square$

*Proof of Proposition 2.1.* Denote  $D = B(x_0, \varepsilon)$  and let  $M = \sup_{y \in D} V(y)$ . By assumption (1) and Lemma 3.3,  $\lim_{y \rightarrow x} p_D(t, x, y) = \infty$  for every  $x \in D$ . Using this we can derive that the transition operator of the process  $(X_t)_{t \geq 0}$  killed in  $D$ , i.e.,  $P_t^D f(x) = \int_D p_D(t, x, y) f(y) dy$ ,  $f \in L^1(D)$ , is not bounded from  $L^1(D)$  to  $L^\infty(D)$ . Since for  $f \geq 0$  we have

$$T_t f(x) \geq \mathbf{E}^x \left[ e^{-\int_0^t V(X_s) ds} f(X_t); t < \tau_D \right] \geq e^{-Mt} \int_D p_D(t, x, y) f(y) dy, \quad x \in D,$$

this clearly means that  $T_t$  is not a bounded operator from  $L^1(\mathbf{R}^d)$  to  $L^\infty(\mathbf{R}^d)$  as well. Thus also  $T_{t/2}$  cannot be a bounded operator from  $L^2(\mathbf{R}^d)$  to  $L^\infty(\mathbf{R}^d)$ .  $\square$

*Proof of Theorem 2.8.* First assume that the potential  $V$  is such that the semigroup  $\{T_t : t \geq 0\}$  is  $t_0$ -GSD. From Corollary 2.3 (2) we directly derive that there is  $R > 0$  such that  $2C_{14}t_0V(x)\nu(x) \geq -\nu(x)\log\nu(x)$ , for all  $x \in B(0, R)^c$ . By integrating in this inequality with respect to Lebesgue measure over an arbitrary Borel set  $A \subset B(0, R)^c$ , we obtain  $F_A^{2C_{14}t_0V}(\nu) \geq 0$ .

Consider the converse implication. Since for some  $t_0 > 0$ ,  $R > 0$  and any Borel set  $A \subset B(0, R)^c$  we have  $\int_A \nu(x)(|\log\nu(x)| - t_0V(x))dx \leq 0$ , the bound  $|\log\nu(x)| \leq t_0V(x)$  holds for almost every  $x \in B(0, R)^c$ . By Assumptions 2.1 (1) and 2.5, it is immediate to deduce that there is  $R_1 > 1$  such that  $|\log\nu(x)| \leq 2C_{14}t_0V(x)$  for all  $|x| > R_1$ . Again, by Corollary 2.3 (2), this implies  $8C_{14}t_0$ -GSD of  $\{T_t : t \geq 0\}$ .  $\square$

**Acknowledgements.** We thank T. Kulczycki for discussions on intrinsic ultracontractivity and eigenfunction estimates. Our special appreciation goes to M. Kwaśnicki for reading a preliminary version of the manuscript, and for many valuable comments and discussions on intrinsic ultracontractivity and estimates of local extrema of harmonic functions. K. Kaleta was supported by the National Science Center (Poland) grant on the basis of the decision no. DEC-2011/03/D/ST1/00311. We both thank IHES, Bures-sur-Yvette, for a visiting fellowship providing an ideal environment to joint work.

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